$K_{2,t}$ MINORS IN DENSE GRAPHS

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JOINT WORK WITH

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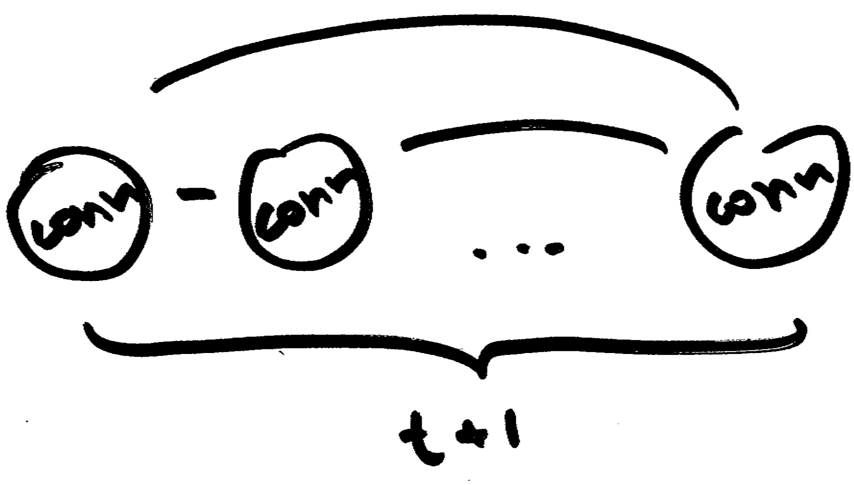
HADWIGER'S CONJ

LET G BE A LOOPLESS GRAPH,
ASSUME G IS NOT t-COLORABLE.
THEN G HAS A K_{t+1} MINOR

($\Leftrightarrow$ A LOOPLESS GRAPH G HAS A $K_{\chi(G)}$ MINOR)



SMALL t :

$t = 1 \Rightarrow$ 

$t = 2 \Rightarrow$ $\triangle$ - MINOR

$t = 3 \Rightarrow$ G IS NOT SERIES-PARALLEL

$\Rightarrow$ K_4 - MINOR

$t = 4 \Rightarrow$ K_5 MINOR

(EQUIVALENT TO 4-CT; WAGNER)

$t = 5 \Rightarrow$ K_6 MINOR

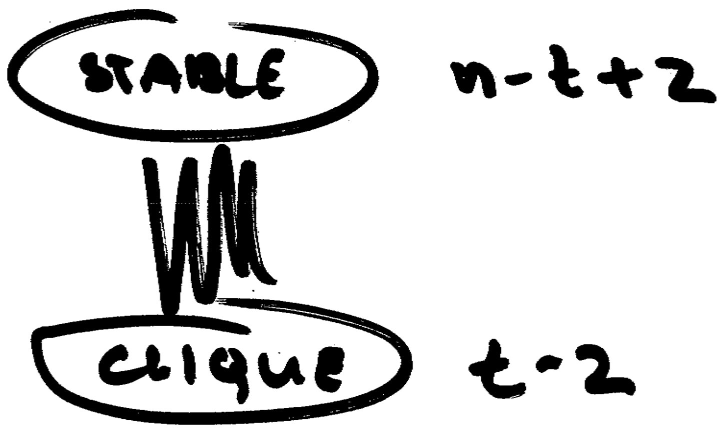
(ROBERTSON, SEYMOUR, THOMAS)

$t \geq 6$ OPEN

A RELATED QUESTION:

ASSUMING G IS SIMPLE, DOES LARGE $|E(G)|/|V(G)|$ FORCE A K_t - MINOR ?

A GRAPH WITH NO K_t-MINOR:



$|V(G)| = n$

$|E(G)| = n(t-2) - \frac{1}{2}(t-2)(t-1)$

FOR SMALL t THIS IS AN EXTREMAL EXAMPLE

FOR SMALL t:

IF $|V(G)| = n$, AND

$$|E(G)| > n(t-2) - \frac{1}{2}(t-2)(t-1)$$

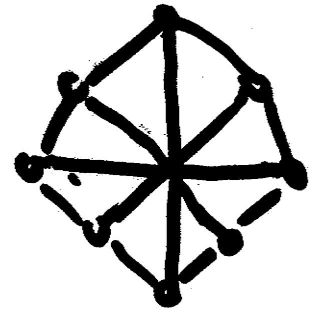
THEN G HAS A K_t - MINOR

$t=2$ $\quad E > 0 \Rightarrow K_2$ - MINOR

$t=3$ $\quad E > n-1 \Rightarrow K_3$ - MINOR

$t=4$ $\quad E > 2n-3 \Rightarrow G$ IS NOT SP $\Rightarrow K_4$ - MINOR

$t=5$ $\quad E > 3n-6 \Rightarrow K_5$ - MINOR

WAGNER: GRAPHS WITH NO K_5 - MINOR ARE OBTAINED FROM PLANAR GRAPHS AND 

BY 1,2,3 - SUMS

MADER: $t=6$ $E > 4n-10 \Rightarrow K_6$-MINOR

$t=7$ $E > 5n-15 \Rightarrow K_7$-MINOR

BUT:

$t=8$ $E > 6n-21 \not\Rightarrow K_8$-MINOR

$K_{10} - \{\text{perfect matching}\}$

$E = 40 > 6 \cdot 10 - 21 = 39$

BUT NO K_8-MINOR

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THM (KOSTOCHKA ; THOMASON)

FOR LARGE t, n

THE MAX # OF EDGES IN A GRAPH ON n VERTICES WITH NO K_t MINOR IS

$$O\left(t\sqrt{\log t}\right) n$$

WHAT ABOUT OTHER MINORS?

THM (MADER)

FOR EVERY GRAPH H THERE EXISTS A CONSTANT C_H, S.T.

IF G IS SIMPLE, AND

$|E(G)| > C_H |V(G)|$

THEN G HAS AN H-MINOR

BIPARTITE MINORS

$K_{1,t}$

$$|E| > \frac{1}{2}(t-1)|V|$$

$$\Downarrow$$

SOME VERTEX HAS DEGREE $\geq t$

$$\Downarrow$$

$K_{1,t}$ MINOR

EXTREMAL EXAMPLE:

$$\underbrace{K_t \quad K_t \quad \cdots \quad K_t}_{n/t}$$

GET A BETTER BOUND IF

G IS CONNECTED:

THM (DING, JOHNSON, SEYMOUR)

G CONNECTED, SIMPLE

$|E(G)| \geq |V(G)| + \frac{1}{2}(t-1)(t-2)$

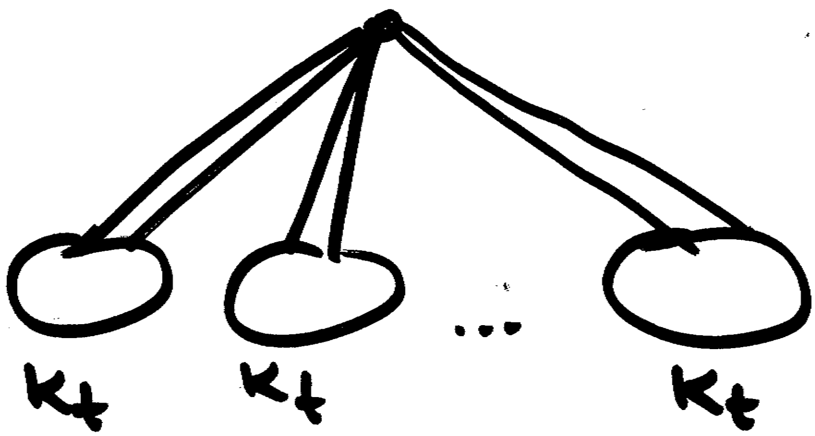
AND $|V(G)| \neq t+2$

$\Downarrow$

G HAS A $K_{1,t}$ -MINOR

$K_{2,t}$

A GRAPH WITH NO $K_{2,t}$ MINOR



$$|E(G)| = \frac{1}{2}(n-1)(t+1)$$

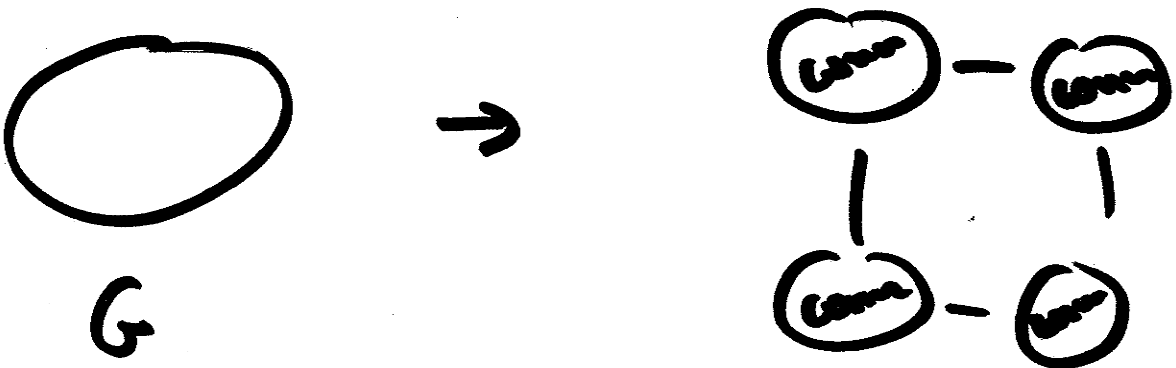
<u>THM 4</u> G SIMPLE. IF

$$|E(G)| \geq \frac{1}{2}(V(G)-1)(t+1)$$

THEN G HAS A $K_{2,t}$ MINOR

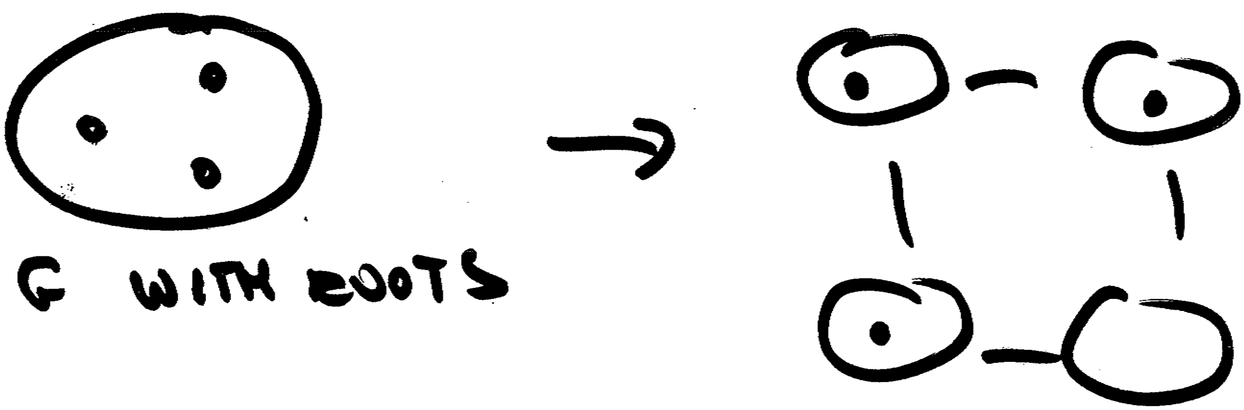
CONJECTURED, AND PROVED FOR $t > 10^{29}$, BY MYERS

ROOTED MINORS



G

C_4 MINOR IN G



G WITH ROOTS

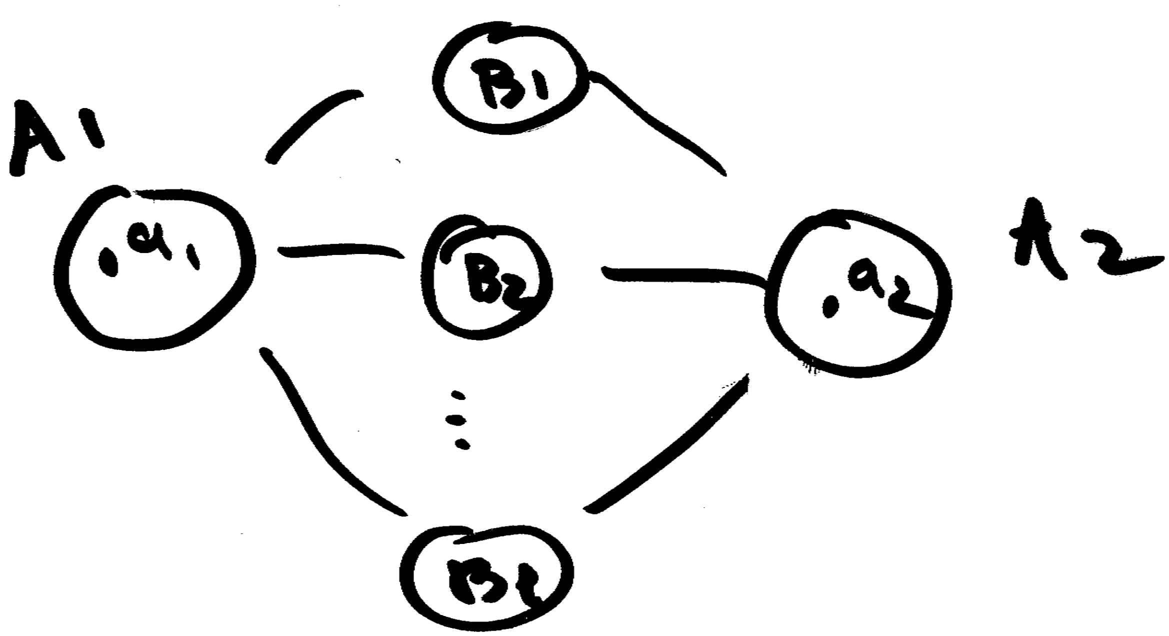
ROOTED C_4 MINOR

PRESCRIBED VERTICES OF G "BECOME" PRESCRIBED VERTICES OF THE MINOR

DEF

LET G BE A GRAPH WITH ROOTS a_1, a_2. G HAS AN (a_1, a_2)-ROOTED $K_{2,t}$ MINOR IF $\exists$ $A_1, A_2, B_1, \ldots, B_t$

- DISJOINT CONNECTED S.G.'S OF G, S.T.
- $a_1 \in V(A_1)$, $a_2 \in V(A_2)$ AND
- $\forall$ i,j THERE IS AN EDGE FROM $V(A_i)$ TO $V(B_j)$



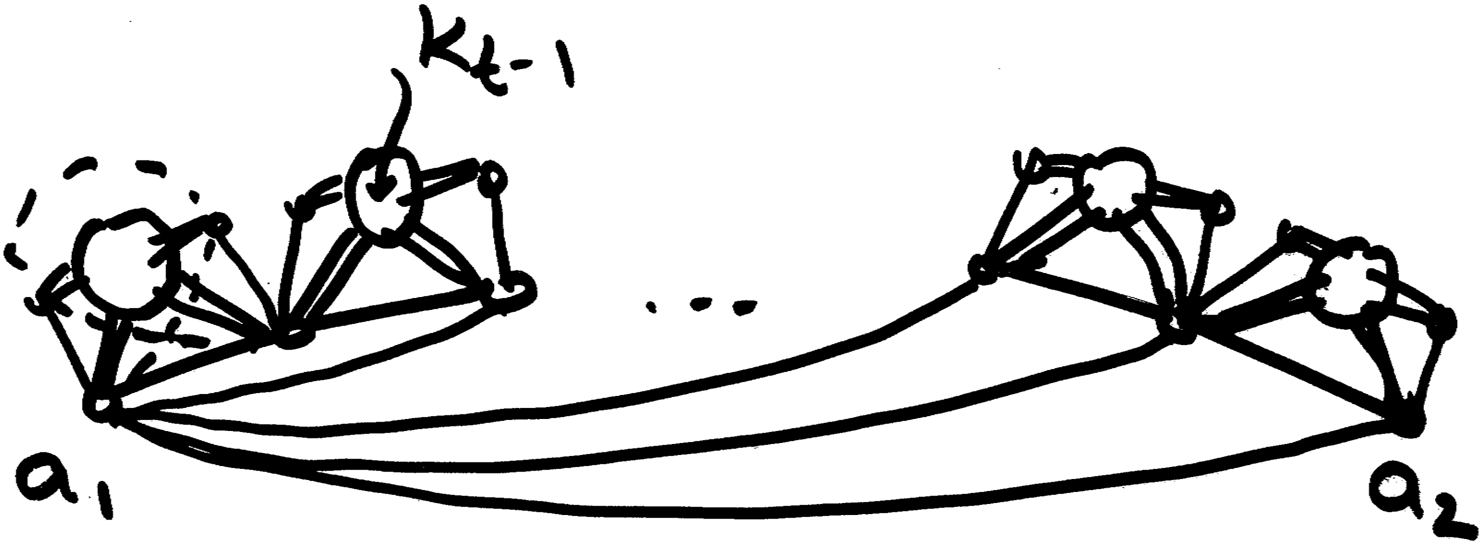
<u>THM 2</u> LET G BE SIMPLE, $a_1, a_2 \in V(G)$. <u>ASSUME</u> THAT EVERY OTHER VERTEX OF G IS 2-CONNECTED TO $\{a_1, a_2\}$.

<u>ASSUME</u>

$$|E(G)| > \frac{1}{2}(|V(G)|-1)\left(t+3-\frac{4}{t+2}\right)-1$$

<u>THEN</u> G HAS AN (a_1,a_2)-ROOTED $K_{2,t}$ MINOR

EXTREMAL EXAMPLE



$$|V(G)| = (t+2)(k-1)+1$$

$$|E(G)| = \frac{1}{2}\left(|V(G)|-1\right)\left(t+3-\frac{4}{t+2}\right)-1$$

NO (a_1, a_2)-ROOTED $K_{2,t}$-MINOR

WHAT ABOUT $K_{s,t}$ MINORS FOR GENERAL s?

THM (KOSTOCHKA & PRINCE)

EVERY GRAPH WITH AVERAGE DEGREE $\geq t + 3s$ HAS A $K^*_{s,t}$-MINOR (ASSUMING $t \gg s$)

AVERAGE DEGREE $t + 3s - 5\sqrt{s}$ IS NOT ENOUGH

THM (KOSTOCHKA & PRINCE)

FOR LARGE t, IF

$|E(G)| \geq \frac{1}{2}(t+3)(n-2)+1$

THEN G HAS A $K_{3,t}$ MINOR

ABOUT THE PROOFS:

<u>THM 2</u> • $a_1, a_2 \in V(G)$

• EVERY OTHER VERTEX 2-CONN TO $\{a_1, a_2\}$

• $|E(G)| > \frac{1}{2}(n-1)\left(t+3-\frac{4}{t+2}\right)-1$

$\Downarrow$

(a_1,a_2)-ROOTED $K_{2,t}$ MINOR

<u>PF</u> CONSIDER A MIN COUNTEREX.

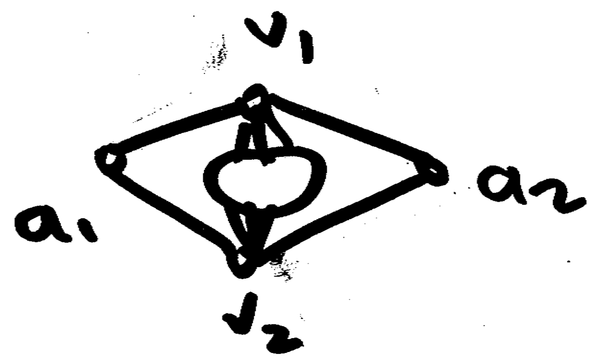
• a_1 IS 2-CONN TO a_2

• EVERY EDGE IS IN $\geq \frac{t+1}{2}$ Δ'S

• EVERY COMMON NBR OF a_1, a_2 HAS DEG $> \frac{n+t+1}{2}$

(DELETE THE COMMON NBR, LOOK FOR $K_{2,t-1}$)

- a_1, a_2 DO NOT HAVE ~~TWO~~ ANY COMMON NBRS



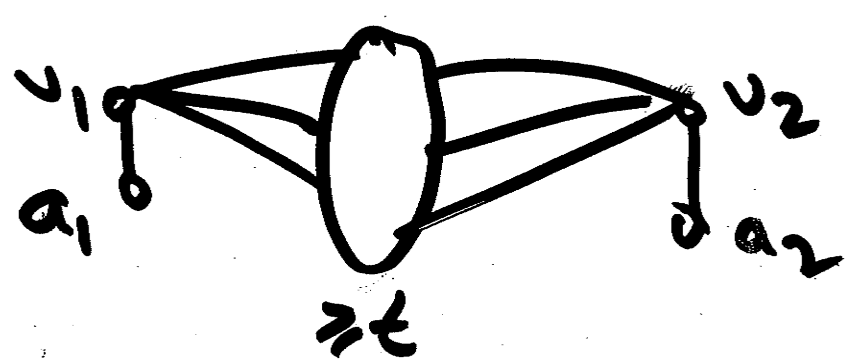
v_1, v_2 EACH HAVE DEG $> \frac{n+t+1}{2}$

$\Downarrow$

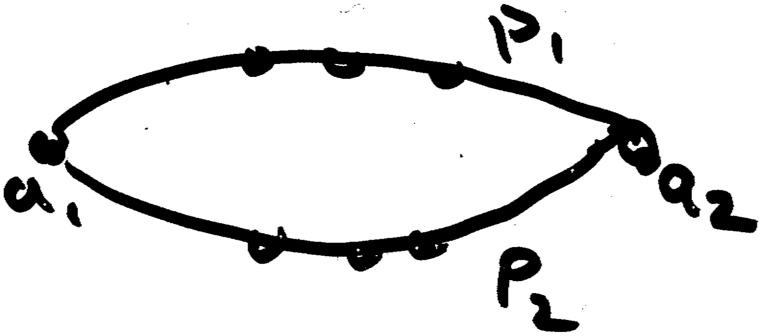
v_1, v_2 HAVE $> t+1$ COMMON NBRS

$\Downarrow$

GET ROOTED $K_{2,t}$



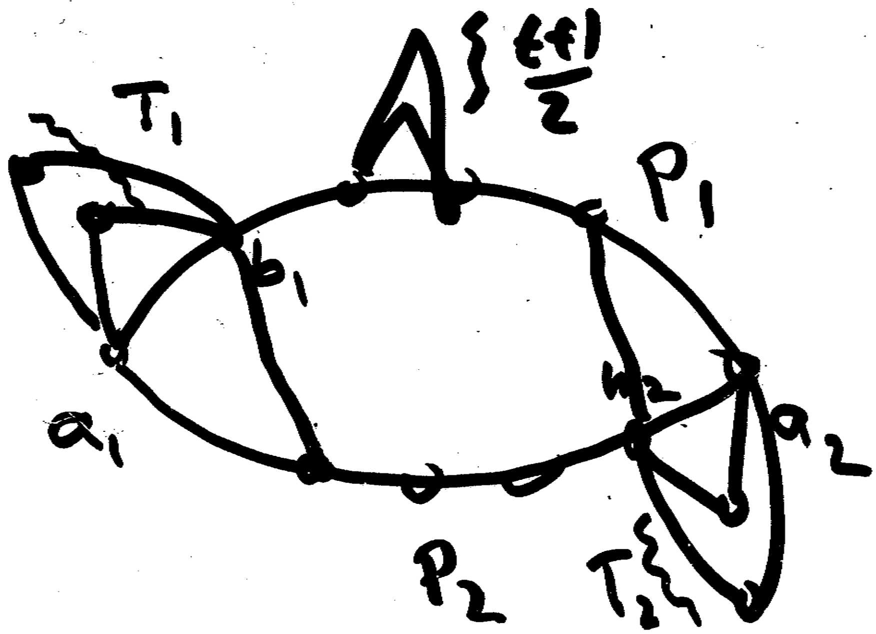
LET P_1, P_2 BE TWO INTERNALLY DISJOINT PATHS BETWEEN a_1, a_2



WITH $P_1 \cup P_2$ MINIMAL

SIMILAR ARGUMENTS SHOW THAT AT LEAST ONE OF P_1, P_2 HAS LENGTH > 4

(OW GET TWO VERTICES OF $P_1 + P_2 \setminus \{a_1, a_2\}$ WITH $\geq t$ COMMON NBRS $\Rightarrow$ (a_1, a_2)-ROOTED $K_{2,t}$)



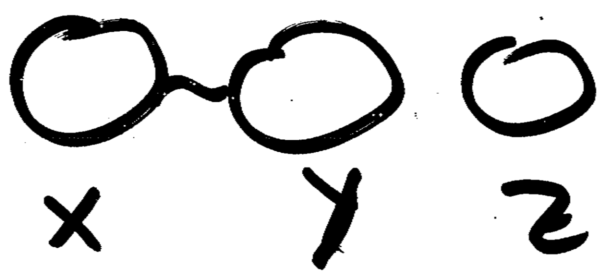
- EACH OF a_1b_1, a_2b_2 IS IN $\geq \frac{t+1}{2}$ TRIANGLES
- $T_1 \cap T_2 = \emptyset$
- ALMOST GET $K_{2,t}$

USE TRIANGLES ON AN INTERNAL EDGE OF P_1

THM 1

$|E| > \frac{1}{2}(n-1)(t+1) \Rightarrow K_{2,t}$ MINOR

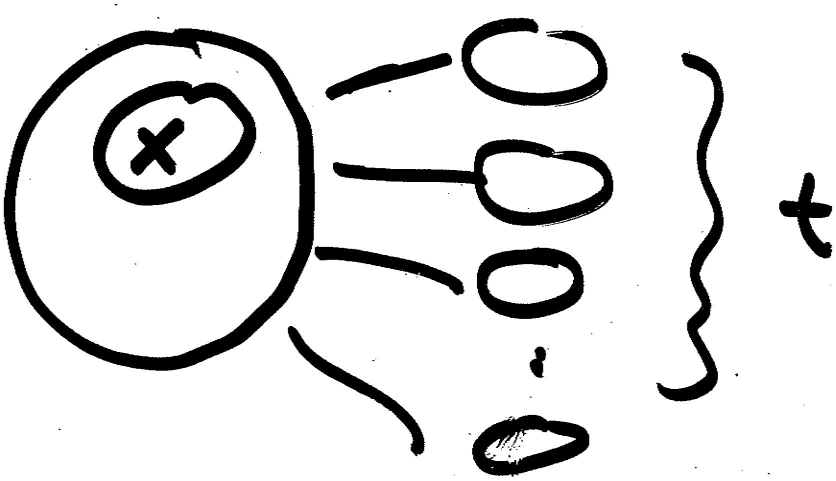
LEMMA



- X CONNECTED
- EVERY $y \in Y$ HAS A NBR IN X
- $|Z| \geq 2(t - |Y|)$

AND ...

THEN THERE IS A $K_{1,t}$ MINOR, S.T.

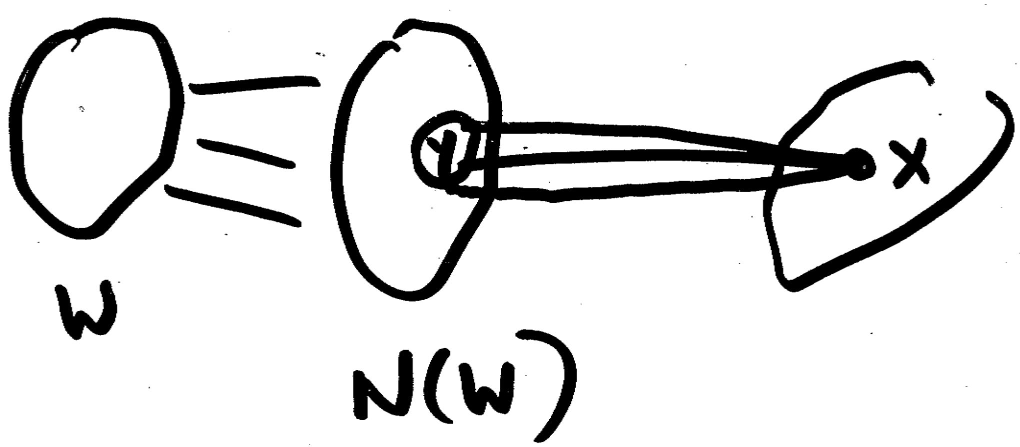


NOW CAN FIND

- A SMALL CONNECTED SET W (≤ 4 VERT) WITH LARGE NEIGHBORHOOD ($\geq t+3$)

AND

- A VERTEX $x \notin W$



S.T. APPLYING THE LEMMA TO $\{x\}$ AND $N(x) \cap N(W)$ IN $G | (N(W) \cup \{x\})$ PRODUCES A $K_{2,t}$ MINOR