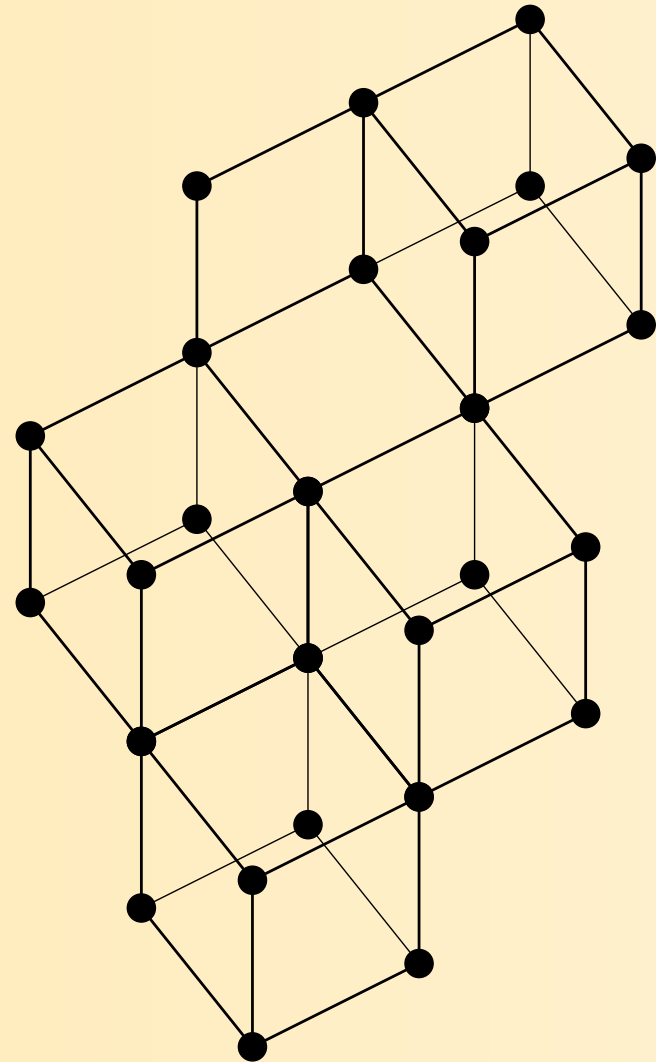


ULD-Lattices: Theory, Instances and Applications

New Directions in ACO
Honoring W.T. Trotter
Atlanta, May 9., 2008

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The Talk

Structure of Schnyder Woods

Lattices from Graphs

Proving Distributivity: ULD-Lattices

Embedded Lattices and D-Polytopes

Contents

Structure of Schnyder Woods

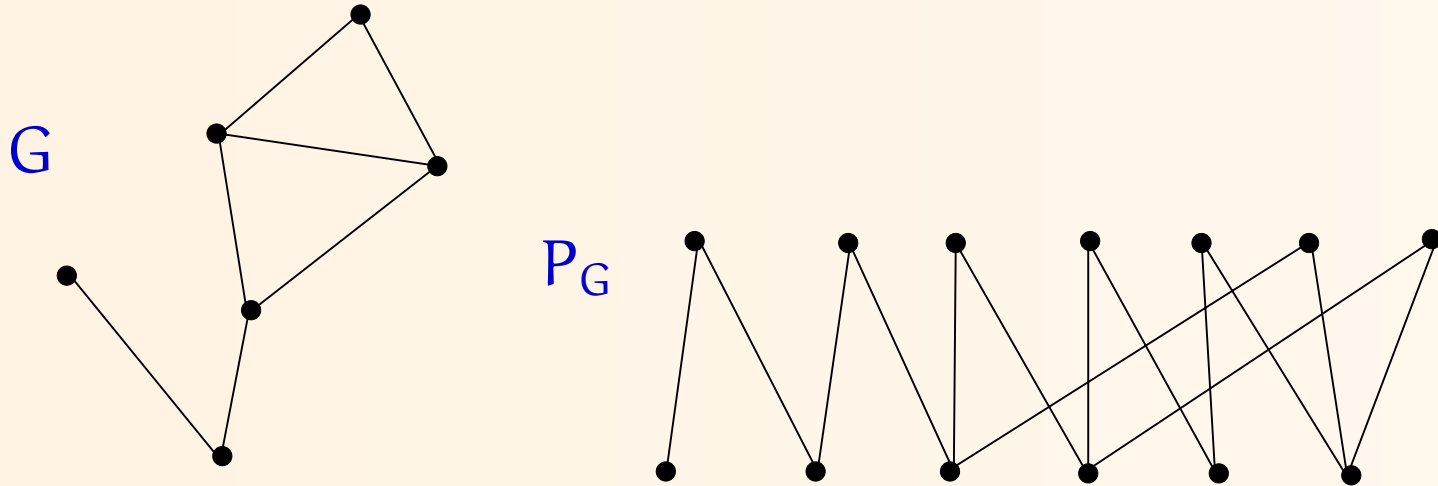
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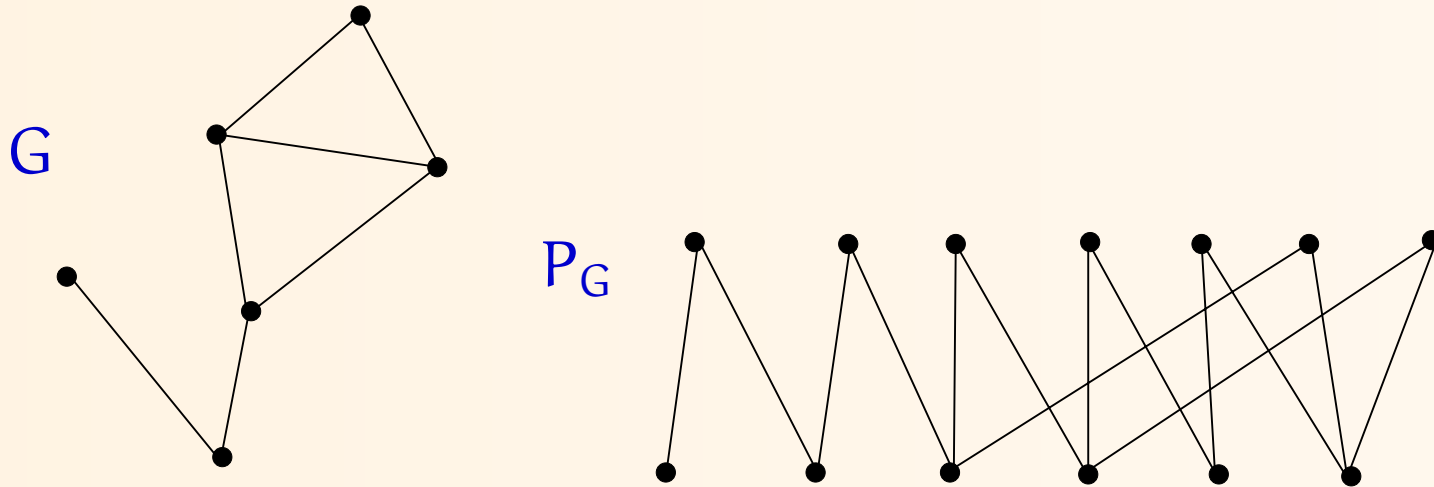
Schnyder's Theorem

The incidence order P_G of a graph G



Schnyder's Theorem

The incidence order P_G of a graph G



Theorem [Schnyder 1989].

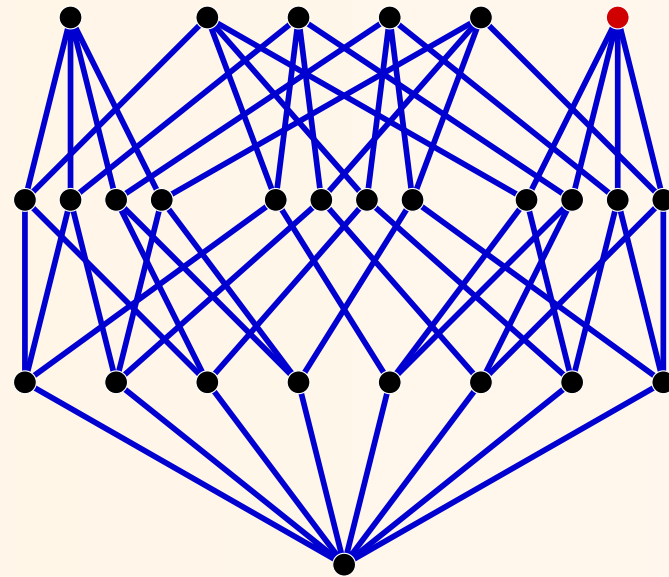
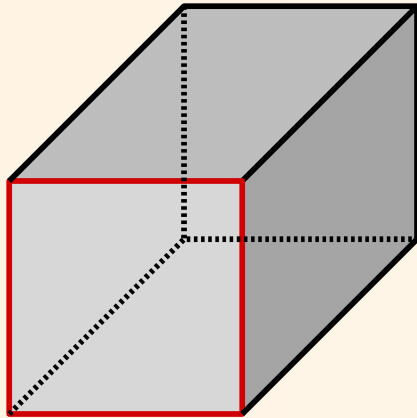
A Graph G is planar $\iff \dim(P_G) \leq 3$.

The Brightwell-Trotter Theorem

Theorem [Brightwell+Trotter '92].

If \mathcal{P} is a 3-polytope and F is any face of \mathcal{P} , then

- $\dim(\mathcal{F}_{\mathcal{P}} \setminus F) = 3$
- $\dim(\mathcal{F}_{\mathcal{P}}) = 4$

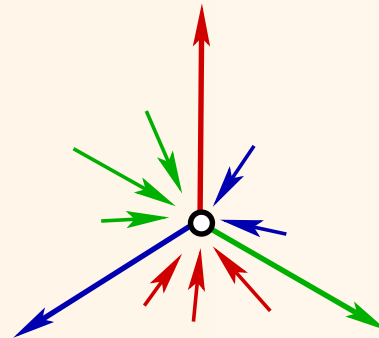


The Tool: Schnyder Woods

G a plane triangulation with outer triangle $F = \{a_1, a_2, a_3\}$.

A coloring and orientation of the interior edges of G with colors 1, 2, 3 is a **Schnyder wood** of G iff

- Inner vertex condition:

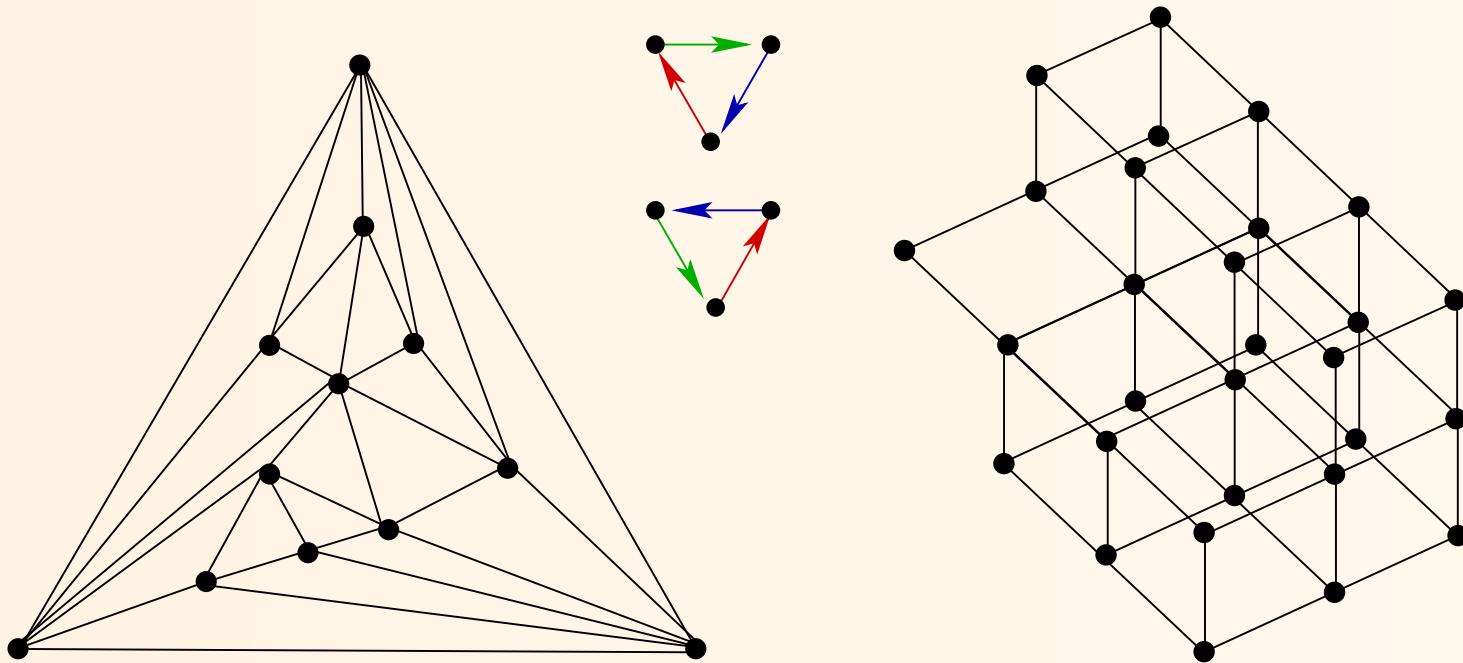


- Edges $\{v, a_i\}$ are oriented $v \rightarrow a_i$ in color i .

↪ Triangle contact representations.

The Lattice of Schnyder Woods

Theorem [de Mendez & Brehm]. The set of Schnyder woods of a plane triangulation G has the structure of a distributive lattice.



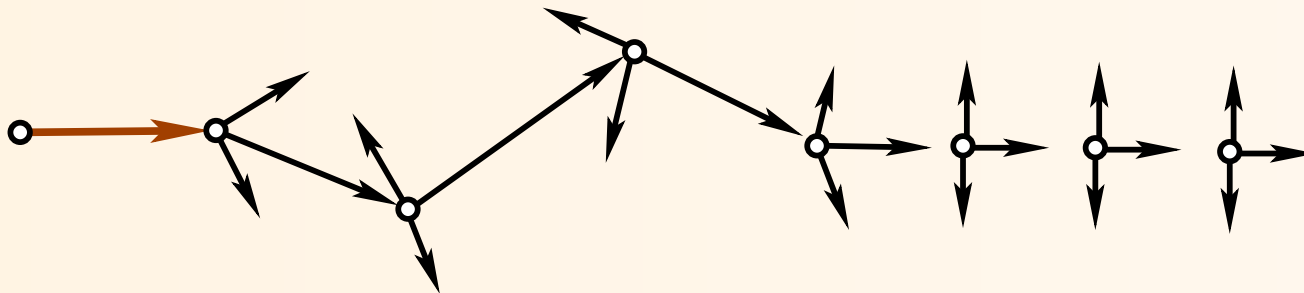
3-orientations

Theorem.

Schnyder woods and 3-orientations are equivalent.

Proof.

- G has $3n - 9$ interior edges and $n - 3$ interior vertices.
 \implies All edges incident to a_i are oriented $\rightarrow a_i$.
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some a_i .

Contents

Structure of Schnyder Woods

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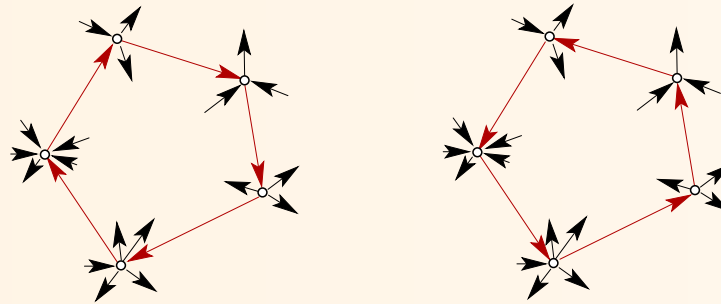
A General Theorem

Definition. Given $G = (V, E)$ and $\alpha : V \rightarrow \mathbb{N}$.
An α -orientation of G is an orientation with
 $\text{outdeg}(v) = \alpha(v)$ for all v .

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Definition. Given $G = (V, E)$ and $\alpha : V \rightarrow \mathbb{N}$.
An α -orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all v .

- Reverting directed cycles preserves α -orientations.



Theorem. The set of α -orientations of a planar graph G has the structure of a distributive lattice.

- Diagram edge \sim revert a directed essential/facial cycle.

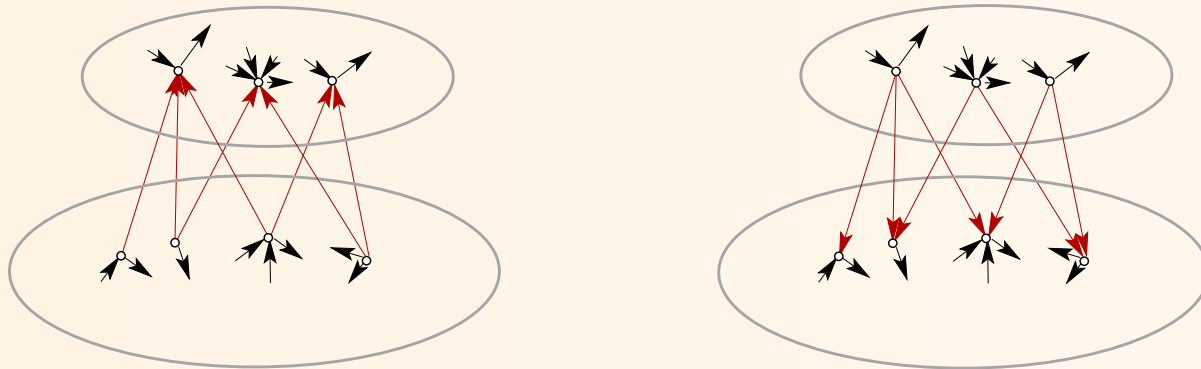
Examples of Lattices

A planar graph has a lattice structure on its

- ↪ Eulerian orientations.
- ↪ spanning trees.
- ↪ matchings and f-factors.
- ↪ Schnyder woods.

A Dual Construction: c-Orientations

- Reorientations of directed cuts preserve **flow-difference** ($\#$ forward arcs $-$ $\#$ backward arcs) along cycles.



Theorem [Propp 1993]. The set of all orientations of a graph with prescribed flow-difference for all cycles has the structure of a distributive lattice.

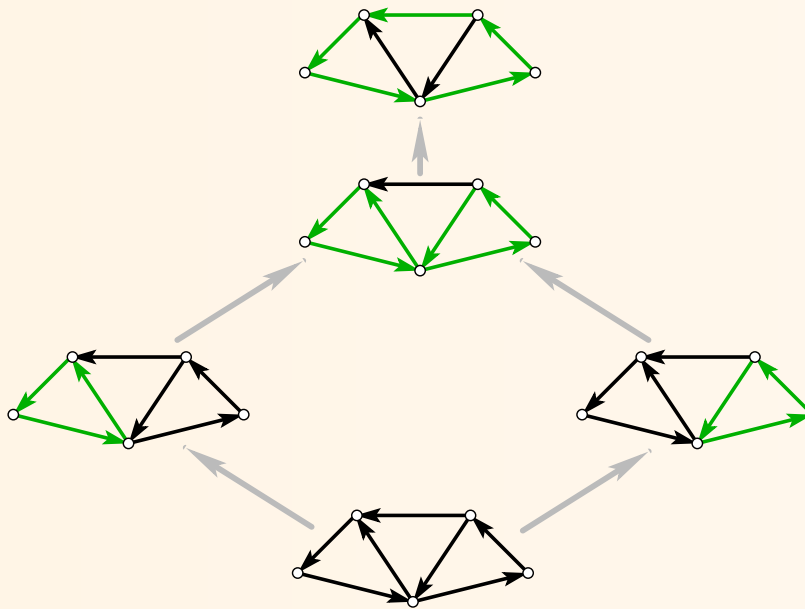
- Diagram edge \sim push a vertex ($\neq v_+$).

Circulations in Planar Graphs

Theorem [Khuller, Naor and Klein 1993].

The set of all integral flows respecting capacity constraints ($\ell(e) \leq f(e) \leq u(e)$) of a planar graph has the structure of a distributive lattice.

$$0 \leq f(e) \leq 1$$



- Diagram edge \sim add or subtract a unit of flow in ccw oriented facial cycle.

Δ -Bonds

$G = (V, E)$ a connected graph with a prescribed orientation.

With $x \in \mathbb{Z}^E$ and C cycle we define the circular flow difference

$$\Delta_x(C) := \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).$$

With $\Delta \in \mathbb{Z}^C$ and $\ell, u \in \mathbb{Z}^E$ let $\mathcal{B}_G(\Delta, \ell, u)$ be the set of $x \in \mathbb{Z}^E$ such that $\Delta_x = \Delta$ and $\ell \leq x \leq u$.

Theorem [Felsner, Knauer 2007]. $\mathcal{B}_G(\Delta, \ell, u)$ is a distributive lattice. The cover relation is vertex pushing.

Δ -Bonds as Generalization

$\mathcal{B}_G(\Delta, \ell, \mathbf{u})$ is the set of $\mathbf{x} \in \mathbb{R}^E$ such that

- $\Delta_{\mathbf{x}} = \Delta$ (circular flow difference)
- $\ell \leq \mathbf{x} \leq \mathbf{u}$ (capacity constraints).

Special cases:

- \mathbf{c} -orientations are $\mathcal{B}_G(\Delta, 0, 1)$
($\Delta(C) = |C^+| - c(C)$).
- Circular flows on planar G are $\mathcal{B}_{G^*}(0, \ell, \mathbf{u})$
(G^* the dual of G).
- α -orientations.

Contents

Structure of Schnyder Woods

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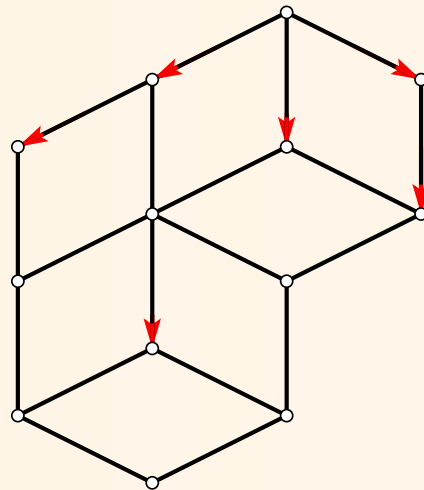
ULD Lattices

Definition. [Dilworth]

A lattice is an **upper locally distributive lattice (ULD)** if each element has a unique minimal representation as meet of meet-irreducibles, i.e., there is a unique mapping $x \rightarrow M_x$

such that

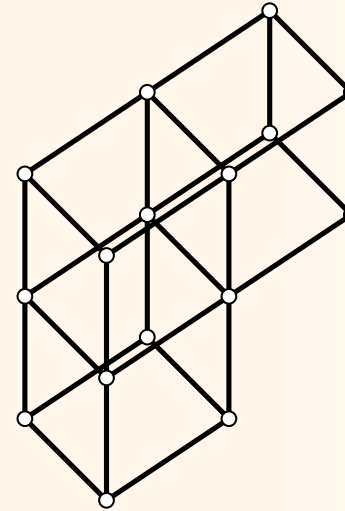
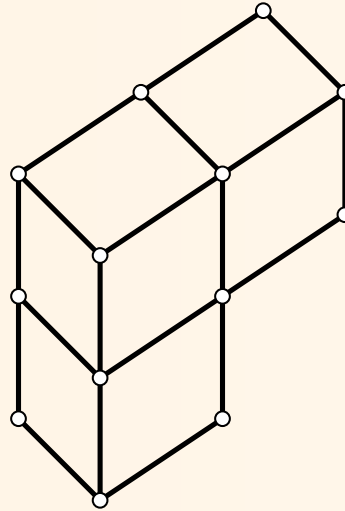
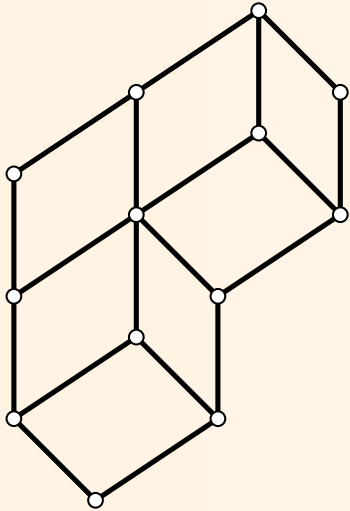
- $x = \bigwedge M_x$ (representation.) and
- $x \neq \bigwedge A$ for all $A \subsetneq M_x$ (minimal).



ULD vs. Distributive

Proposition.

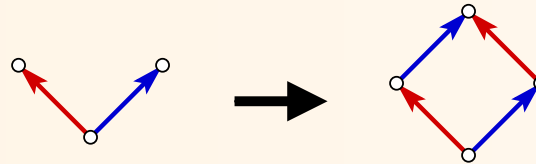
A lattice is distributive \iff it is ULD and LLD.



Diagrams of ULD lattices: A Characterization

A coloring of the edges of a digraph is a **U-coloring** iff

- arcs leaving a vertex have different colors.
- completion property:

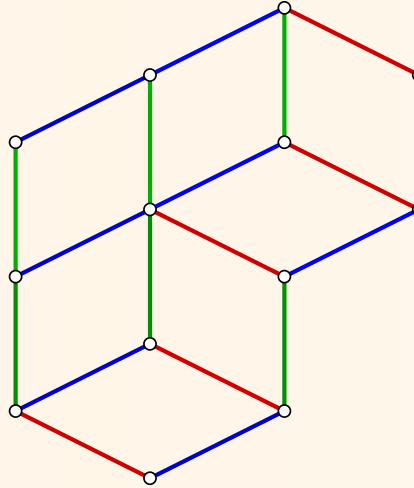


Theorem. A digraph D is acyclic, has a unique source and admits a **U-coloring** $\iff D$ is the diagram of an ULD lattice.

\hookrightarrow Unique **1**.

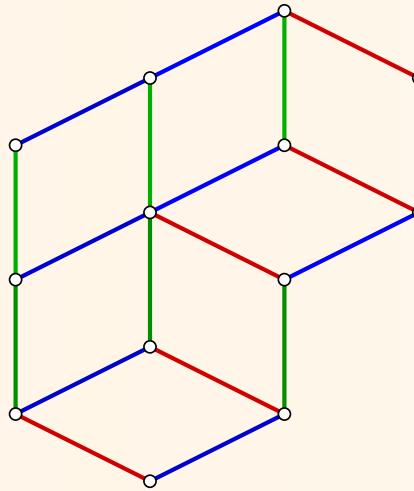
Examples of U-colorings

↪ Classical examples.



Examples of U-colorings

→ Classical examples.



- Chip firing game with a fixed starting position (the source), colors are the names of fired vertices.
- Δ -bond lattices, colors are the names of pushed vertices. (Connected, unique **0**).

Contents

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Embedded Lattices

A \mathcal{U} -coloring of a distributive lattice L yields a cover preserving embedding $\phi : L \rightarrow \mathbb{Z}^{\#\text{colors}}$.

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In the case of Δ -bond lattices there is a polytope $\mathbf{P} = \text{conv}(\phi(\mathbf{L}))$ in \mathbb{R}^{n-1} such that

$$\phi(\mathbf{L}) = \mathbf{P} \cap \mathbb{Z}^{n-1}$$

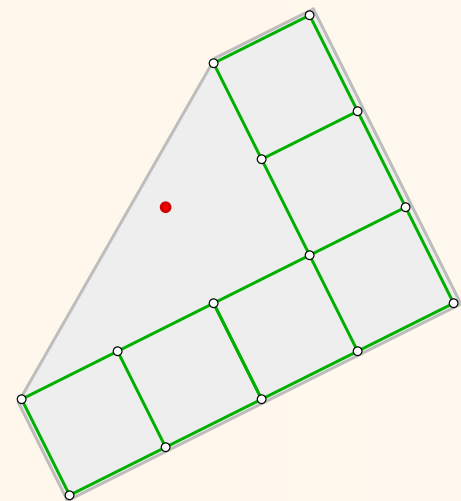
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- This is a special property:



D-Polytopes

Definition. A polytope P is a D -polytope if with $x, y \in P$ also $\max(x, y), \min(x, y) \in P$.

- A D -polytope is a distributive lattice.
- Integral points in a D -polytope form a distributive lattice.

D-Polytopes

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Theorem. A polytope \mathcal{P} is a \mathcal{D} -polytope iff every facet inducing hyperplane of \mathcal{P} is a \mathcal{D} -hyperplane, i.e., closed under \max and \min .

D-Polytopes

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Theorem. A polytope P is a D -polytope iff every facet inducing hyperplane of P is a D -hyperplane, i.e., closed under \max and \min .

Theorem. A hyperplane is a D -hyperplane iff it has a normal $e_i - \lambda_{ij}e_j$ with $\lambda_{ij} \geq 0$.

A First Graph Model for D-Polytopes

Consider $\ell, \mathbf{u} \in \mathbb{Z}^m$ and a Λ -weighted network matrix \mathbf{N}_Λ of a connected graph. (Rows of \mathbf{N}_Λ are of type $\mathbf{e}_i - \lambda_{ij}\mathbf{e}_j$ with $\lambda_{ij} \geq 0$.)

- [Strong case, $\text{rank}(\mathbf{N}_\Lambda) = n$]
The set of $\mathbf{p} \in \mathbb{Z}^n$ with $\ell \leq \mathbf{N}_\Lambda^\top \mathbf{p} \leq \mathbf{u}$ is a distributive lattice.
- [Weak case, $\text{rank}(\mathbf{N}_\Lambda) = n - 1$]
The set of $\mathbf{p} \in \mathbb{Z}^{n-1}$ with $\ell \leq \mathbf{N}_\Lambda^\top(0, \mathbf{p}) \leq \mathbf{u}$ is a distributive lattice.

A Second Graph Model for D-Polytopes

(Rows of N_Λ are of type $\mathbf{e}_i - \lambda_{ij}\mathbf{e}_j$ with $\lambda_{ij} \geq 0$.)

Theorem [Felsner, Knauer 2008].

Let $Z = \ker(N_\Lambda)$ be the space of Λ -circulations. The set of $\mathbf{x} \in \mathbb{Z}^m$ with

- $\underline{\ell} \leq \mathbf{x} \leq \underline{\mathbf{u}}$ (capacity constraints)
- $\langle \mathbf{x}, \mathbf{z} \rangle = 0$ for all $\mathbf{z} \in Z$
(weighted circular flow difference).

is a distributive lattice $\mathcal{D}_G(\Lambda, \underline{\ell}, \underline{\mathbf{u}})$.

- Lattices of Δ -bonds are translates of the case $\lambda_{ij} = 1$.

The Strong Case

For a cycle C let

$$\gamma(C) := \prod_{e \in C^+} \lambda_e \prod_{e \in C^-} \lambda_e^{-1}.$$

A cycle with $\gamma(C) \neq 1$ is strong.

Proposition. $\text{rank}(N_\Lambda) = n$ iff it contains a strong cycle.

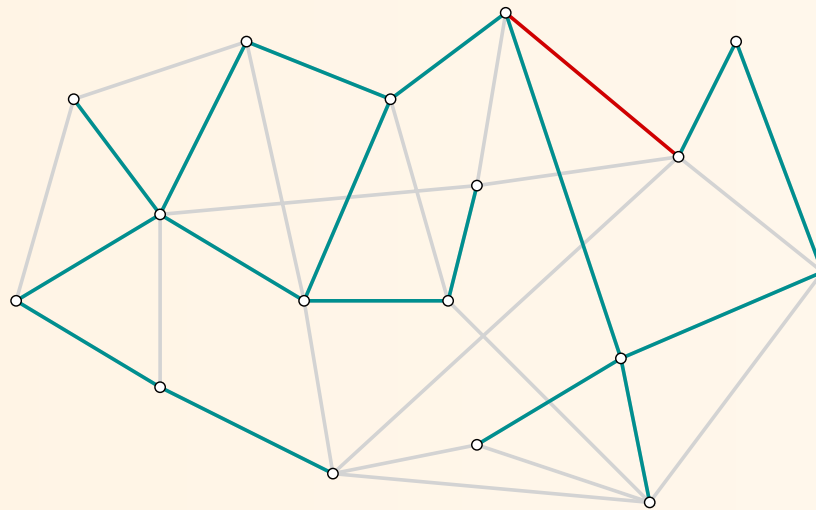
Remark.

C strong \implies there is no circulation with support C .

Fundamental Basis

A fundamental basis for the space of Λ -circulations:

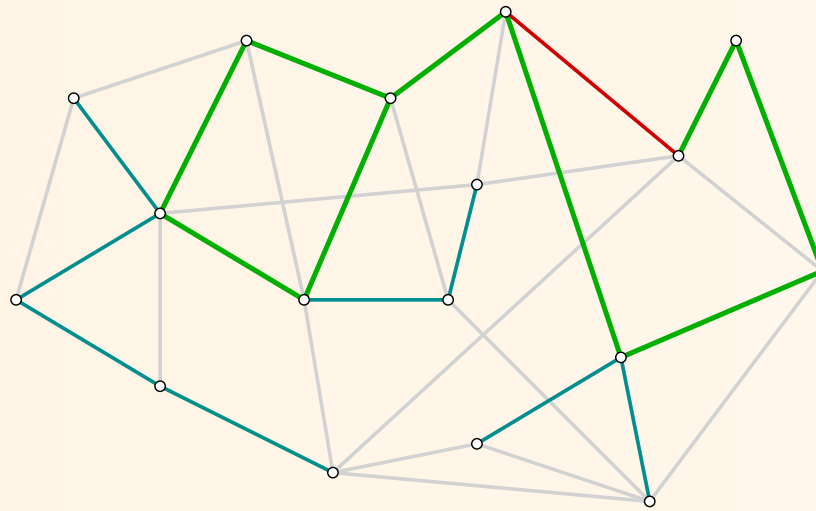
- Fix a 1-tree T , i.e, a unicyclic set of n edges. With $e \notin T$ there is a circulation in $T + e$



Fundamental Basis

A fundamental basis for the space of Λ -circulations:

- Fix a 1-tree T , i.e, a unicyclic set of n edges. With $e \notin T$ there is a circulation in $T + e$



Conclusion

- We now understand the ‘old’ examples of distributive lattices from graphs.
- The **D**-polytope model seems to have potential in various directions.

Conclusion

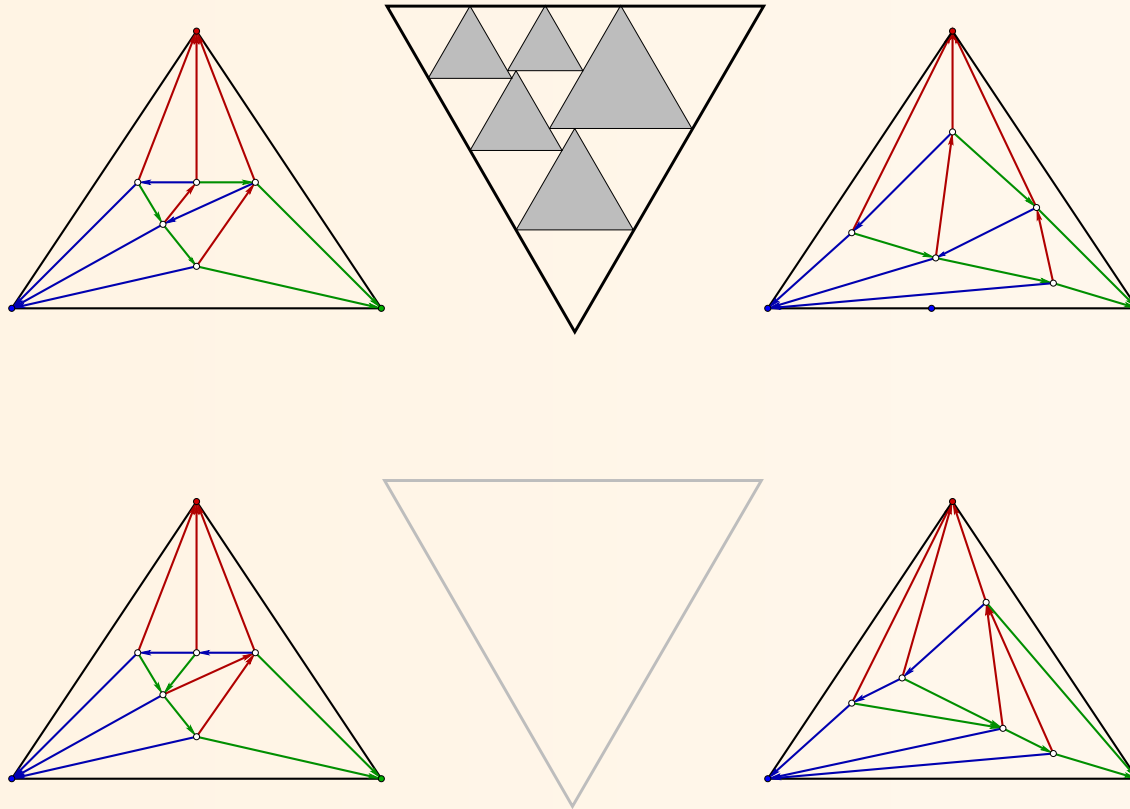
- We now understand the ‘old’ examples of distributive lattices from graphs.
- The **D**-polytope model seems to have potential in various directions.

Hints or comments?

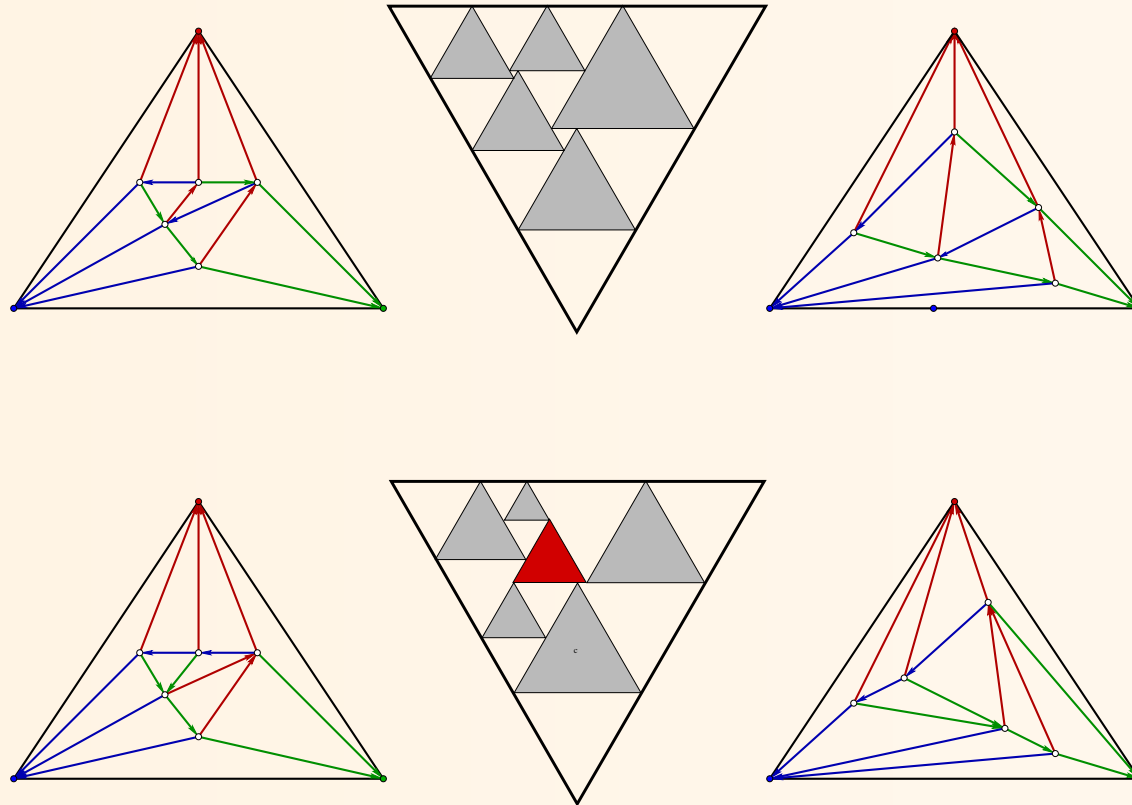
I will be grateful.

THE END

Triangle Contact Representations



Triangle Contact Representations



↪ Back.

Classical Examples

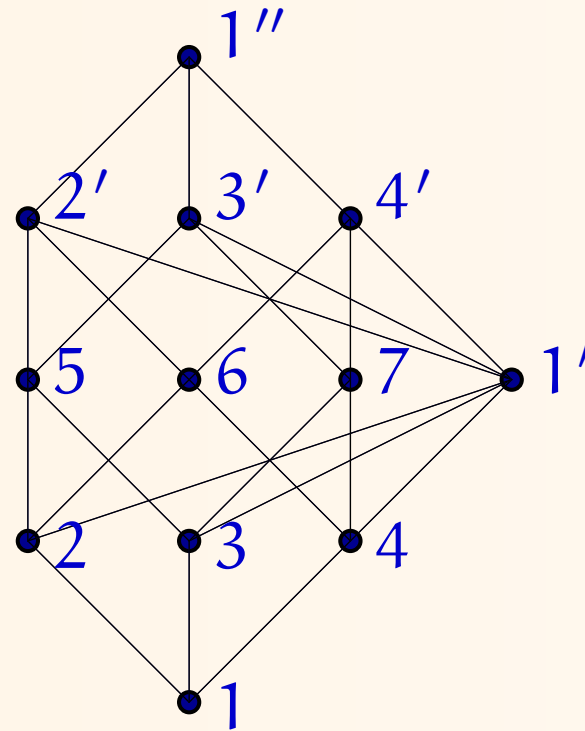
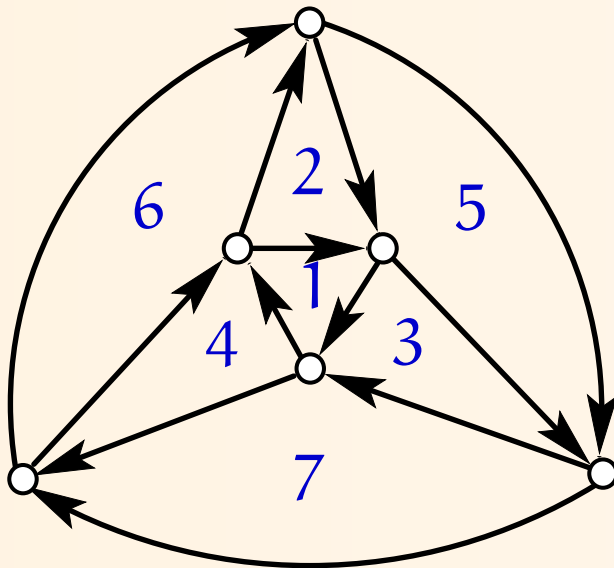
Some LLD lattices with respect to inclusion order:

- Subtrees of a tree (Boulaye '67).
- Convex subsets of posets (Birkhoff and Bennett '85).
- Convex subgraphs of acyclic digraphs (Pfaltz '71).
(C is convex if with x, y all directed (x, y) -paths are in C).
- Convex sets of an abstract convex geometry, this is an universal family of examples (Edelman '80).

↪ [Back](#).

Eulerian Orientations

- Orientations with $\text{outdeg}(v) = \text{indeg}(v)$ for all v ,
i.e. $\alpha(v) = \frac{d(v)}{2}$

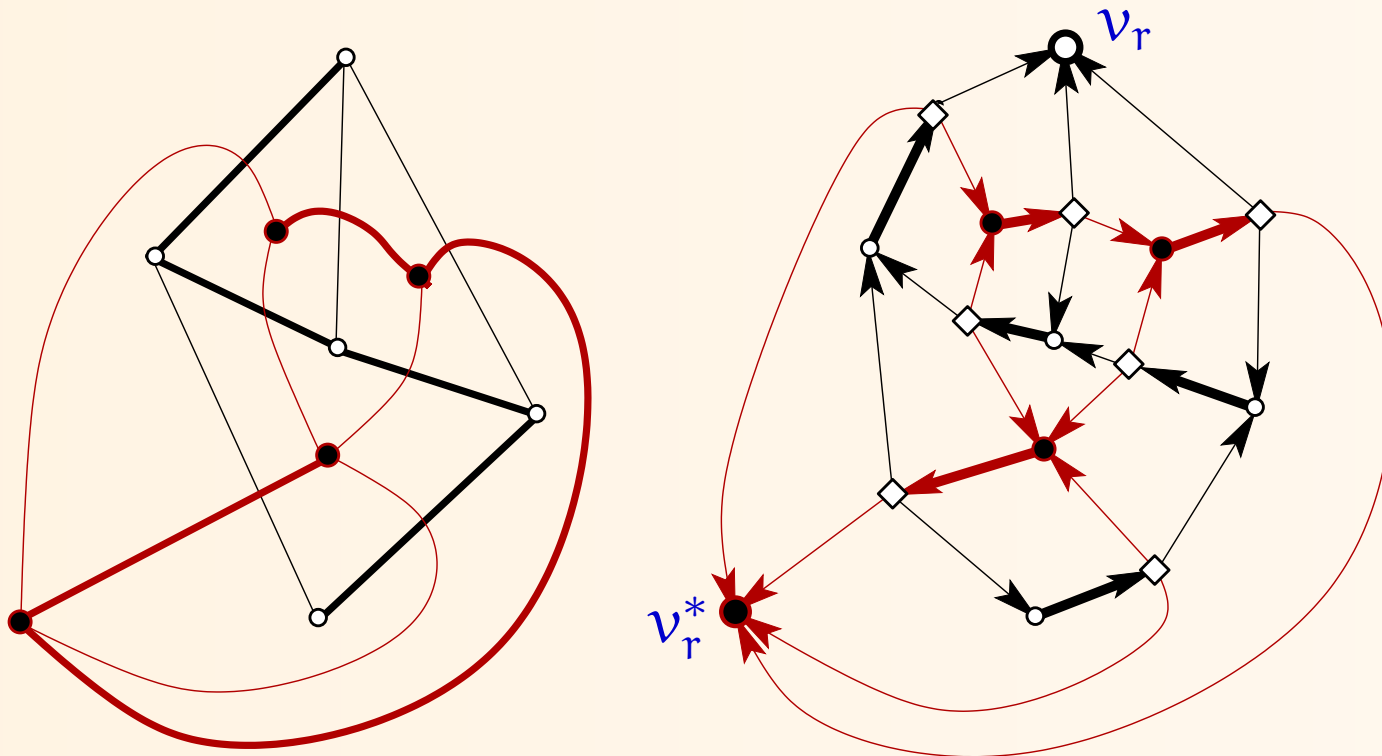


↪ Back.

Spanning Trees

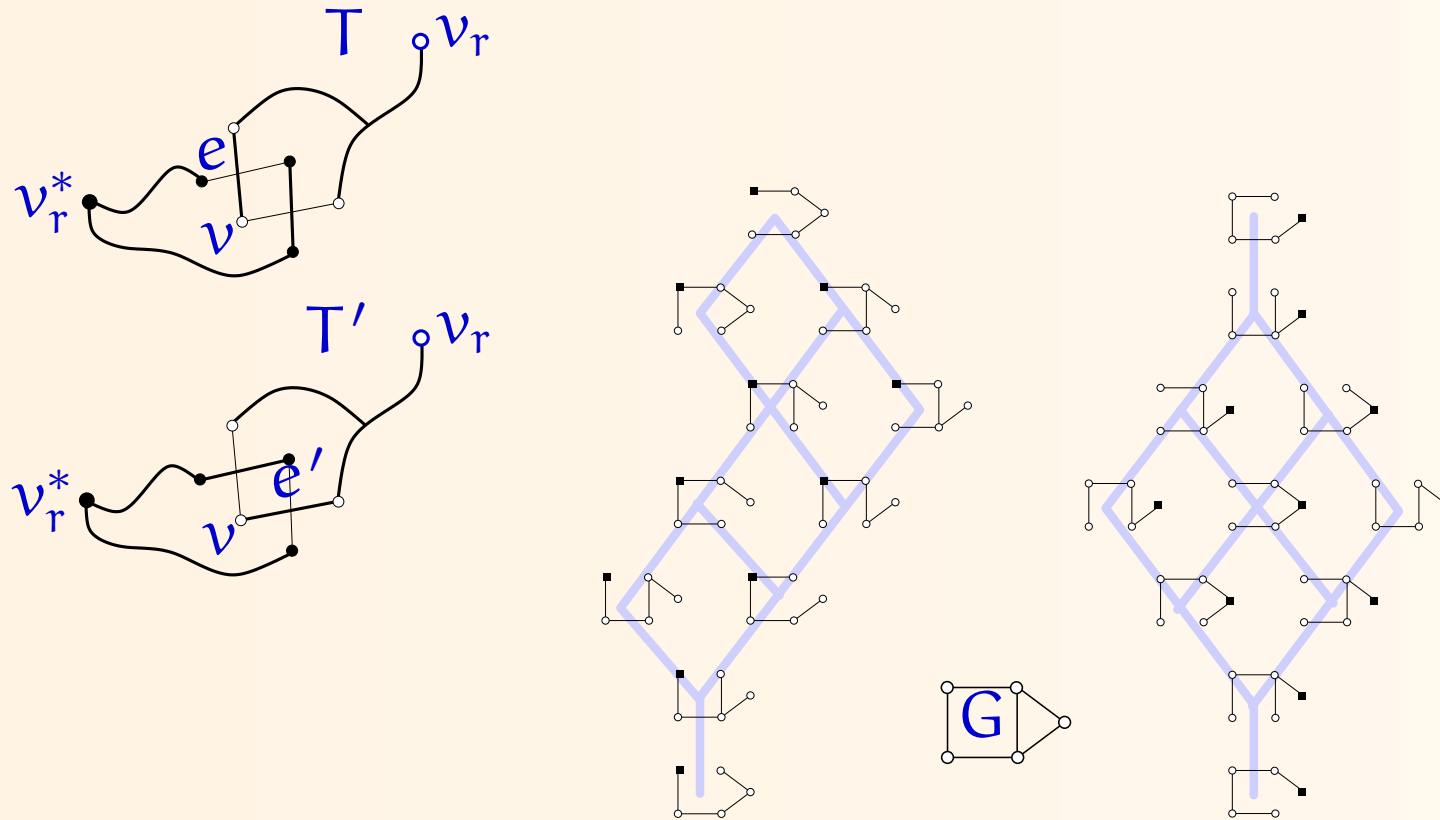
Spanning trees are in bijection with α_T orientations of a rooted primal-dual completion \tilde{G} of G

- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



Lattice of Spanning Trees

Gilmer and Litheland 1986, Propp 1993



Question. How does a change of roots affect the lattice?

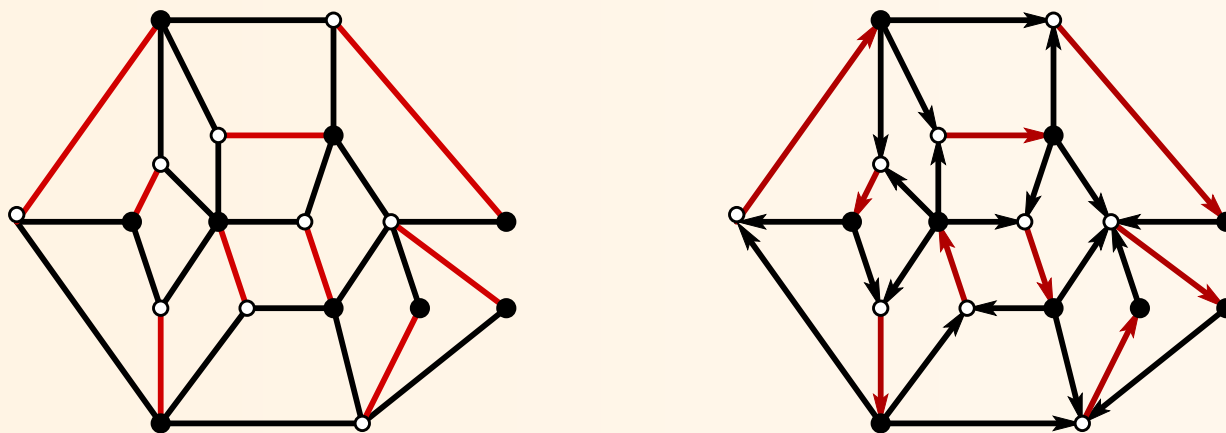
↪ [Back.](#)

Matchings and f-Factors

Let G be planar and bipartite with parts (U, W) . There is a bijection between f -factors of G and α_f orientations:

- Define α_f such that $\text{indeg}(u) = f(u)$ for all $u \in U$ and $\text{outdeg}(w) = f(w)$ for all $w \in W$.

Example. A matching and the corresponding orientation.

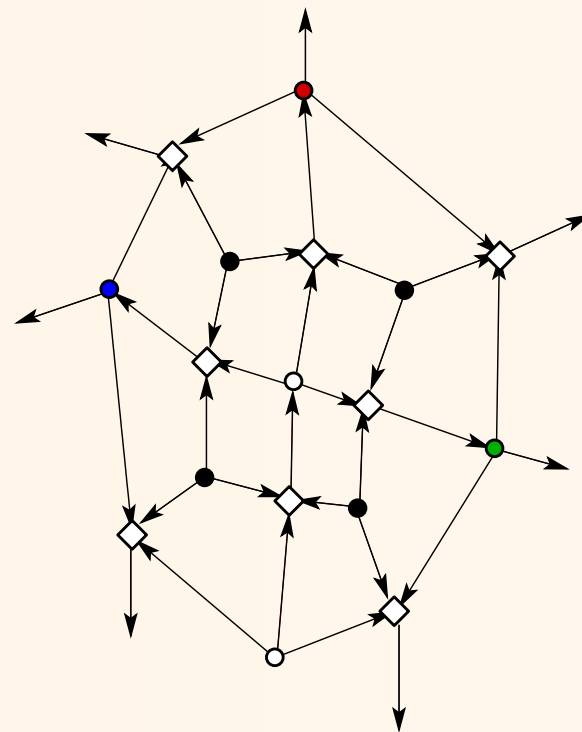
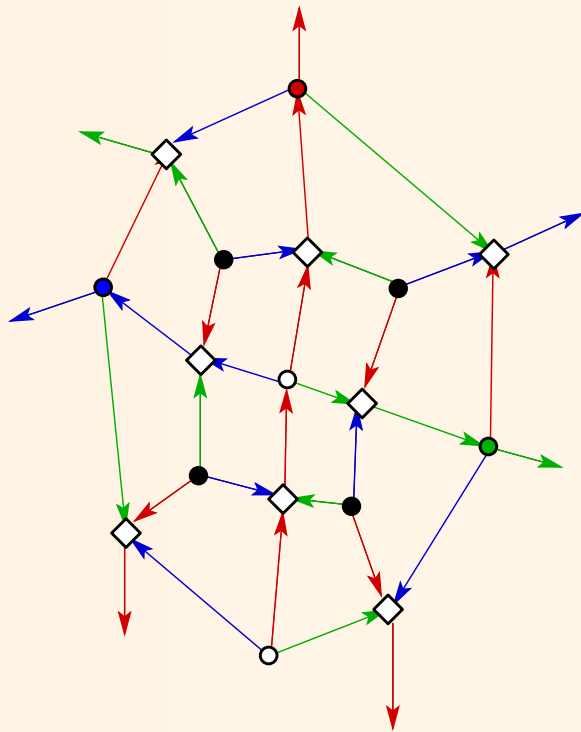


↪ [Back.](#)

Schnyder Woods

Schnyder woods are in bijection with 3-orientations

- $\alpha_S(v) = 3$ for a vertex v and $\alpha_S(v_e) = 1$ for an edge-vertex v_e and exterior rays.



↪ Back.