# On Sperner's Lemma and Scarf's Lemma

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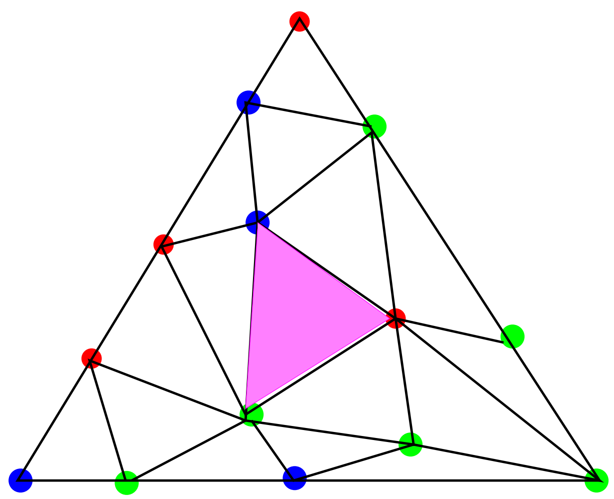
# Sperner's Lemma

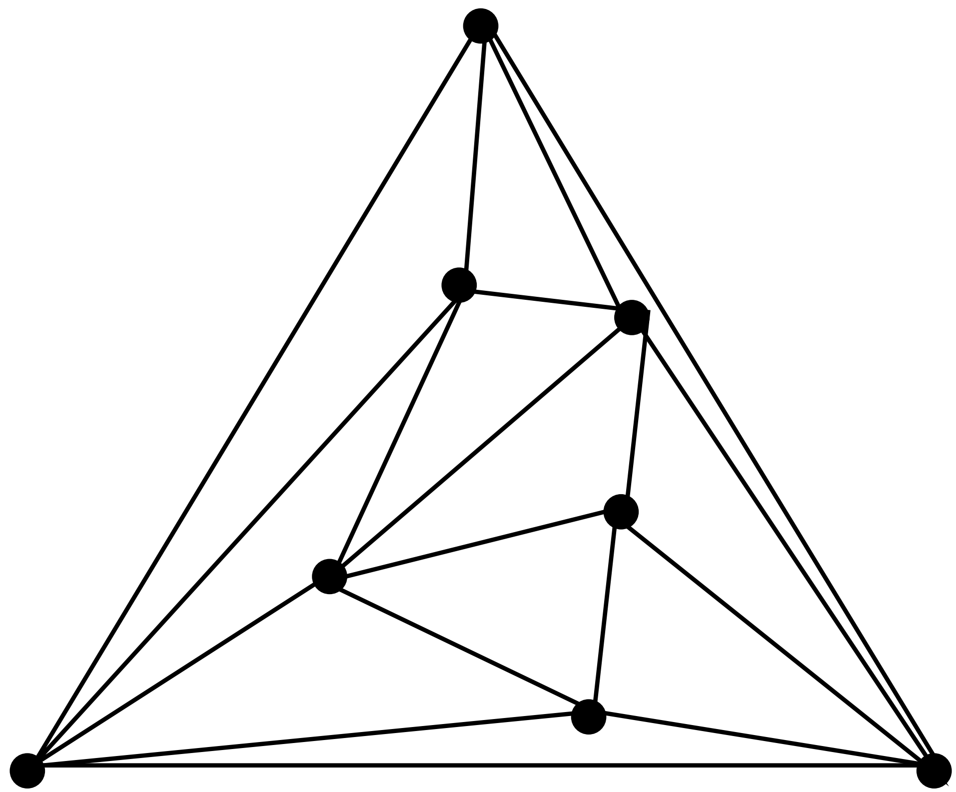
Consider a triangulation of the $n$-dimensional simplex $T$.

Consider any Sperner colouring of the points with $n+1$ colours.

Then the number of multicoloured elementary simplices is ODD. (In particular there is at least one.)

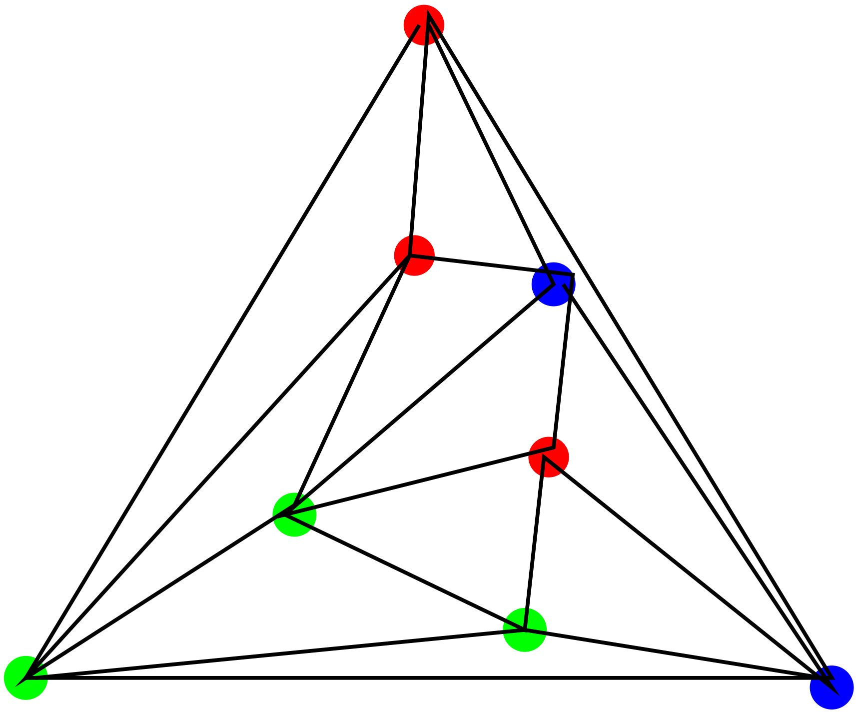


# Restricted Triangulations

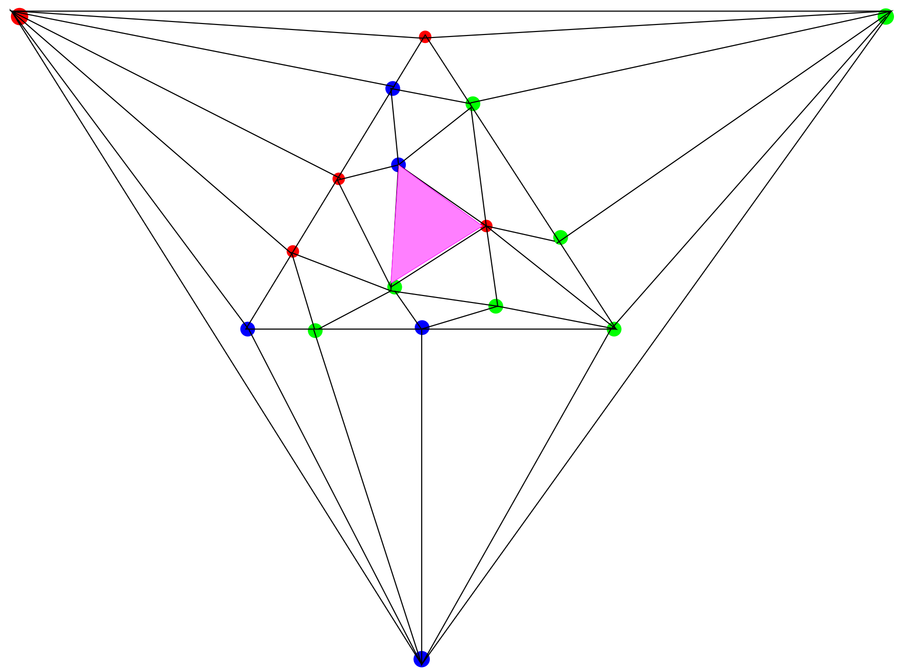


## Sperner's Lemma for Restricted Triangulations

The number of multicoloured elementary simplices in an $(n+1)$-colouring of a restricted triangulation is EVEN.



Sperner’s Lemma for restricted triangulations implies the usual formulation of Sperner’s Lemma:

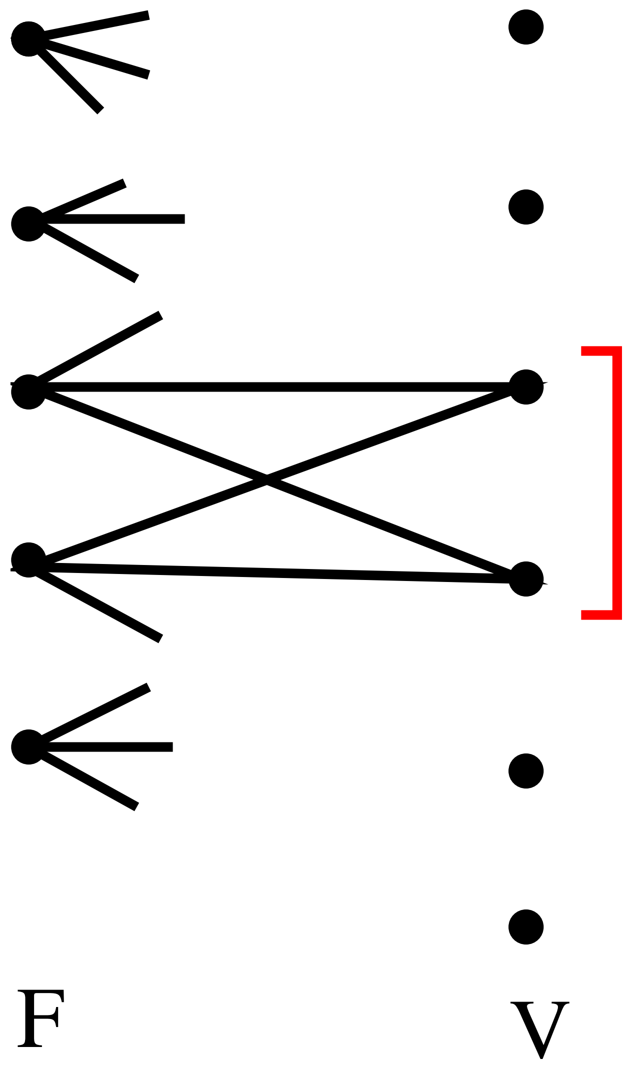


The Sperner colouring guarantees that the only “extra” multicoloured simplex added by this construction is the exterior one.

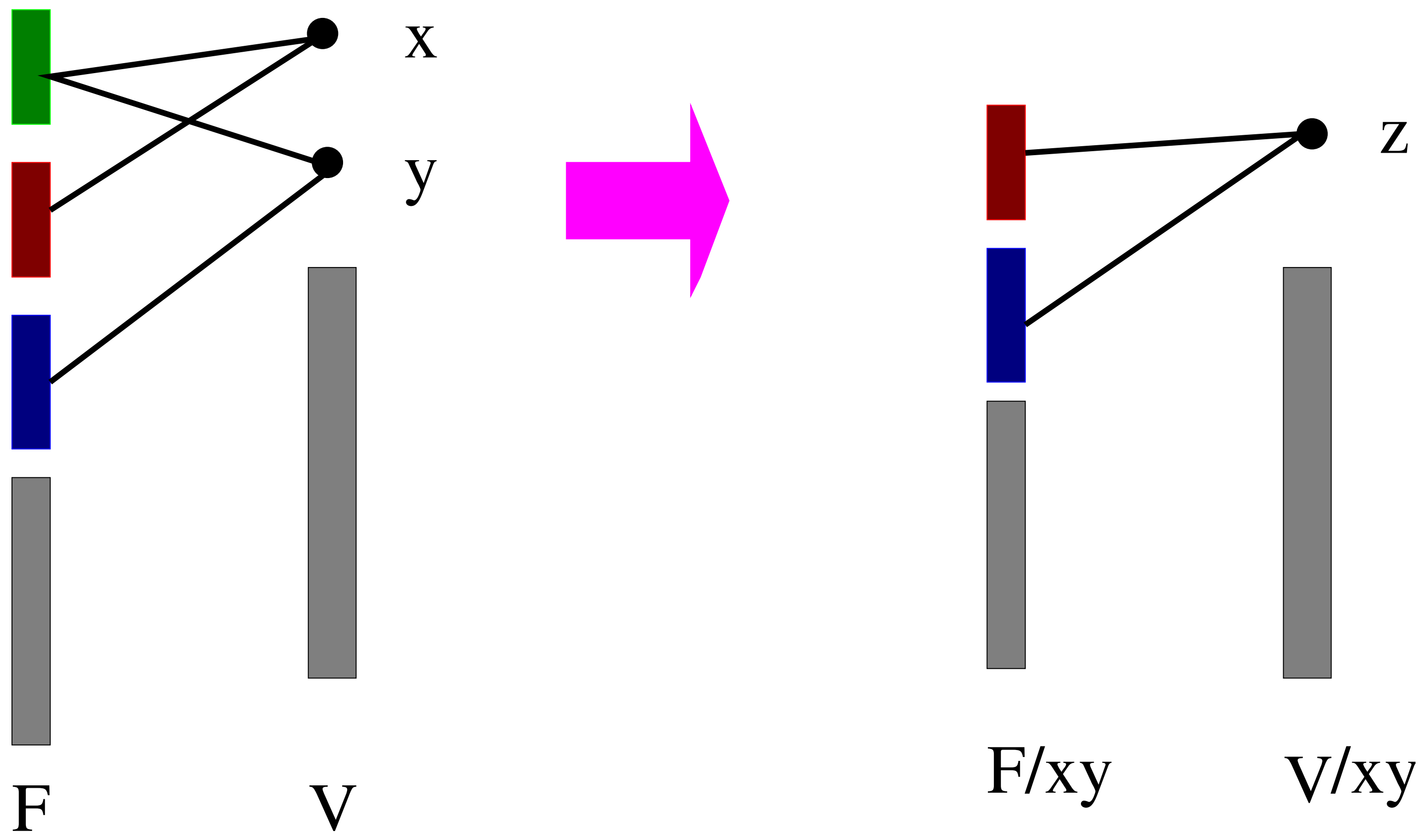
# Bipartite graphs

We say a bipartite graph $G$ with vertex classes $F$ and $V$ is $m$-even if

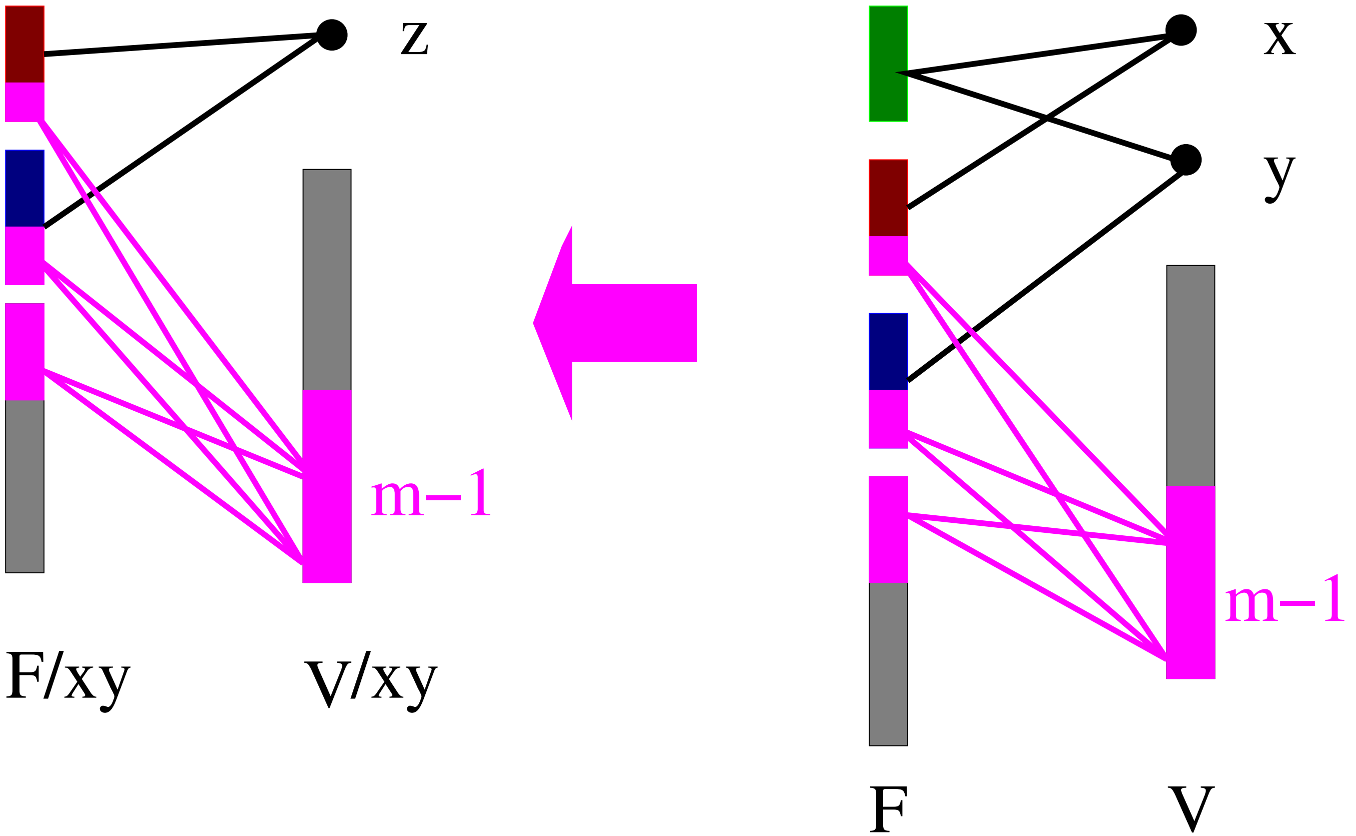
- every $f \in F$ has degree $m$, and
- every $(m-1)$-set $S$ of vertices of $V$ has even codegree, meaning $|\bigcap_{s \in S} \Gamma(s)|$ is even.

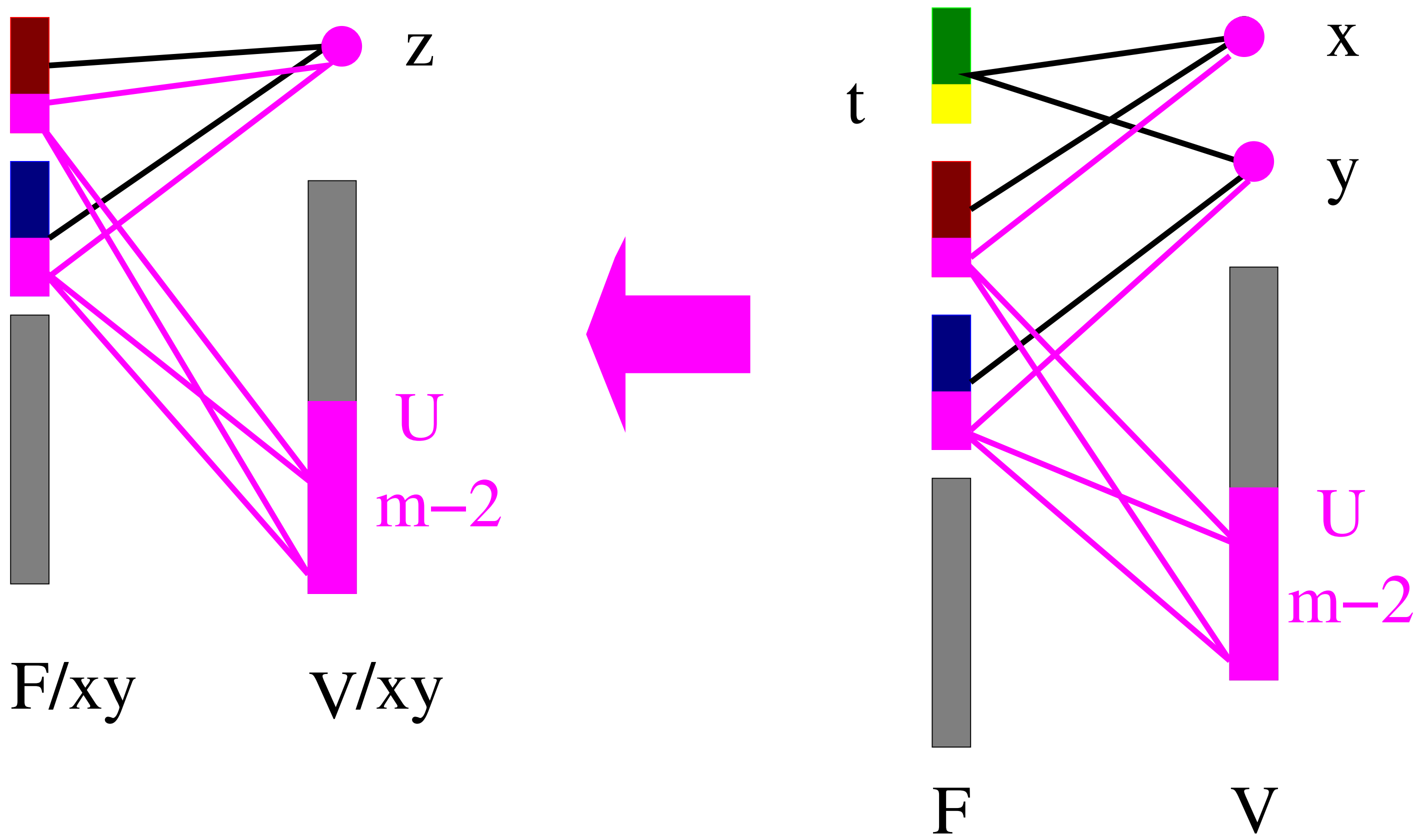

F
V

# An operation on $m$-even graphs



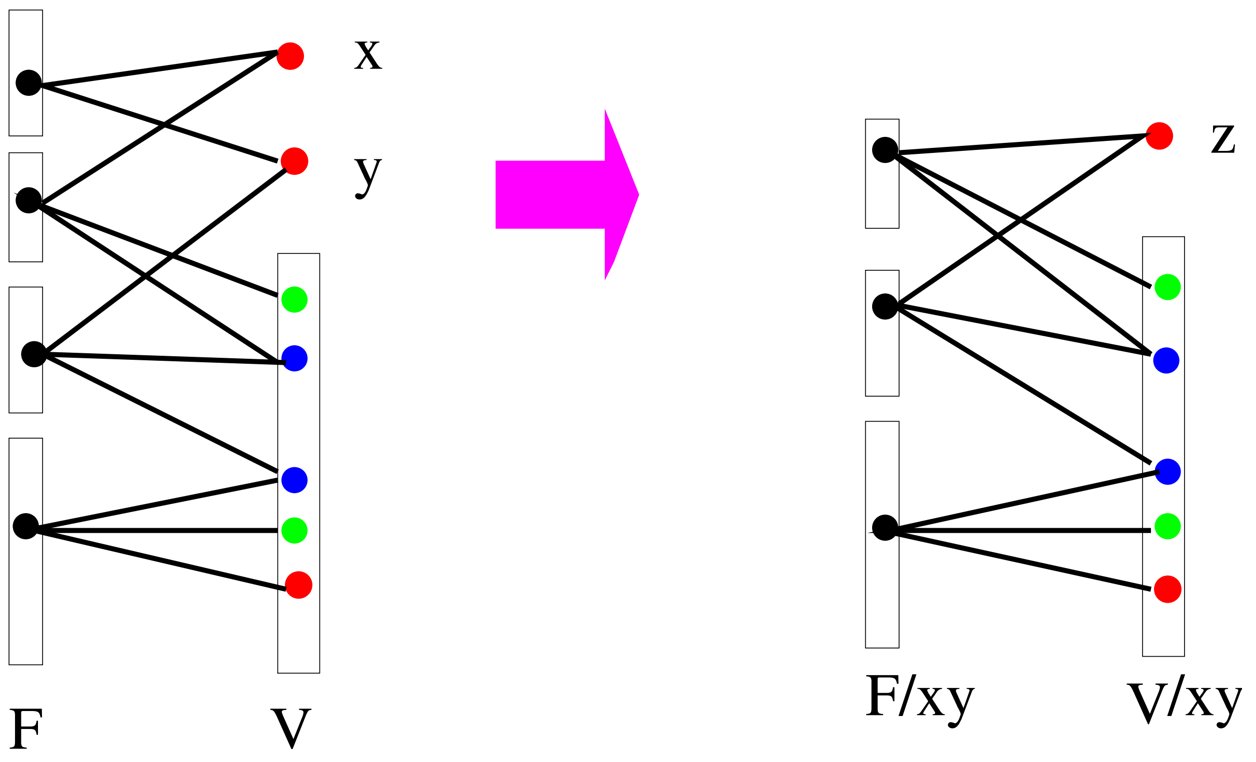
# If $G$ is an $m$-even graph then $G/xy$ is $m$-even



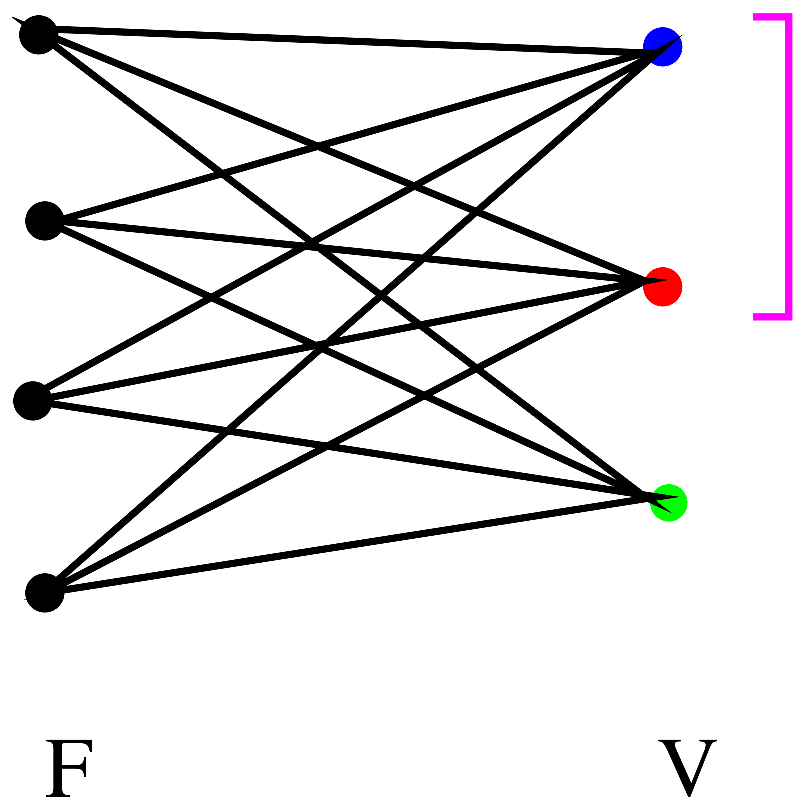



$$|\bigcap_{s\in U\cup\{z\}}\Gamma(s)| = \Big(|\bigcap_{s\in U\cup\{x\}}\Gamma(s)| - t\Big) + \Big(|\bigcap_{s\in U\cup\{y\}}\Gamma(s)| - t\Big).$$

Suppose the vertices $V$ are coloured with $r$ colours. We say a vertex $f \in F$ is multicoloured if all its neighbours are different colours. If $x$ and $y$ have the same colour $c$, and we colour $z$ with $c$, then the number $mc(G)$ of multicoloured vertices of $F$ in $G$ satisfies $mc(G) = mc(G/xy)$.



Suppose $G$ is an $m$-even graph and the vertices $V$ are coloured with $m$ colours. If $|V| = m$ then $mc(G)$ is even.



## Conclusion

LEMMA: If $G$ is an $m$-even graph and the vertices of $V$ are coloured with $m$ colours then $mc(G)$ is EVEN.

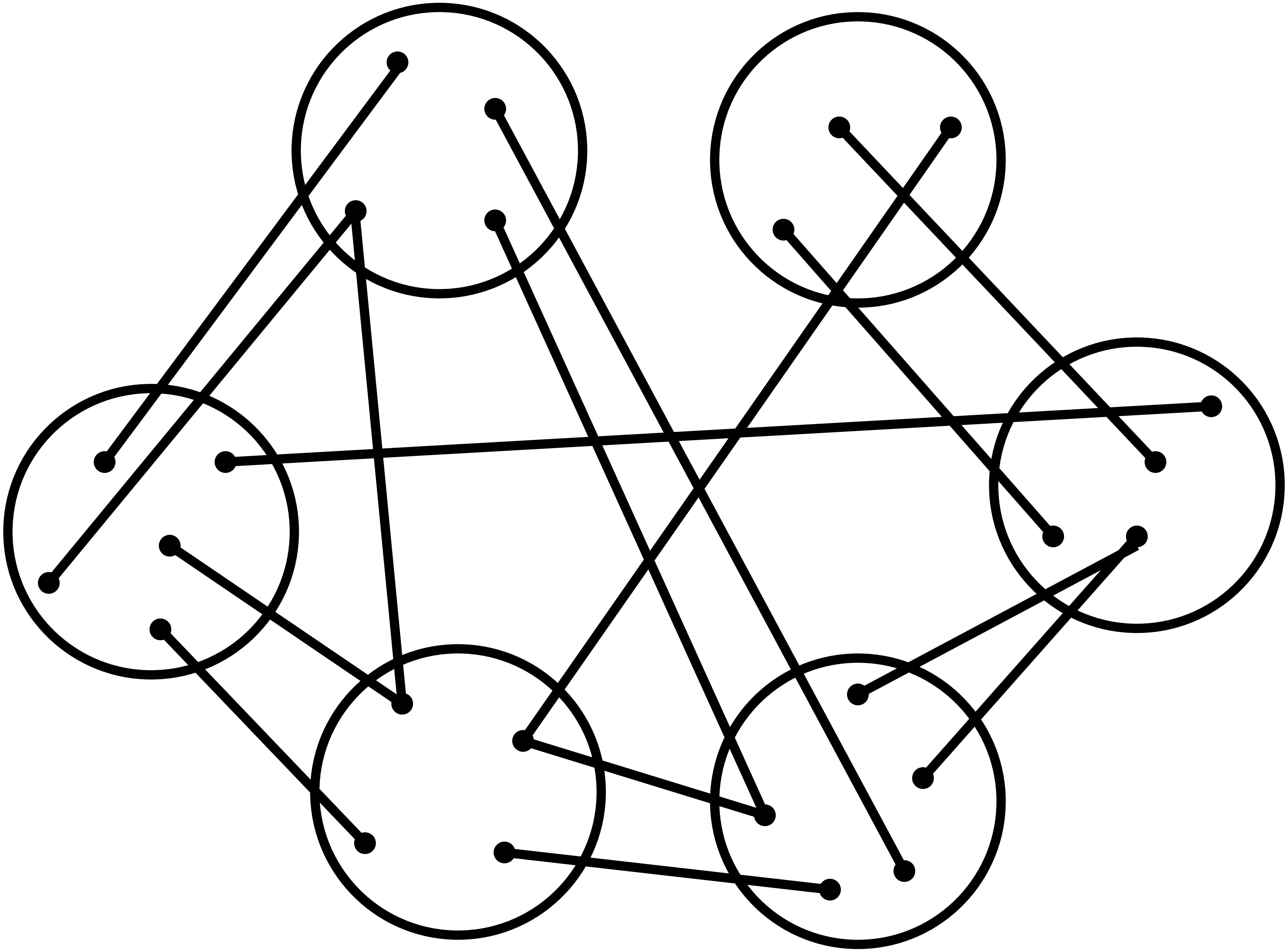
# Sperner's Lemma for Restricted Triangulations

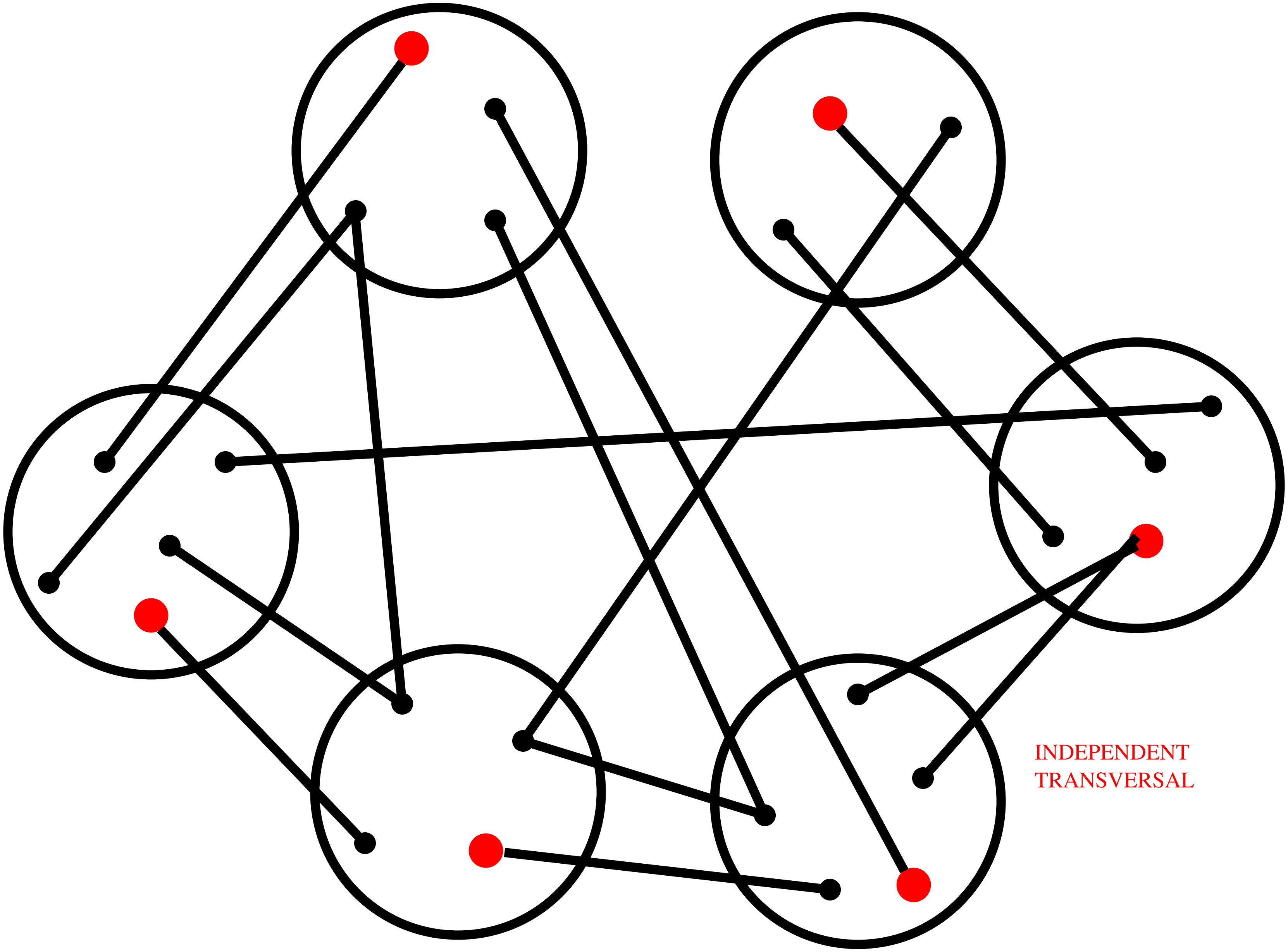
Let $T$ be a restricted triangulation of the $n$-simplex. Let $V$ be the set of points of $T$. For $m = n + 1$ let $F$ be the set of $m$-subsets of points that form the vertices of a full-dimensional elementary simplex. Form a bipartite graph $G$ with vertex sets $V$ and $F$ by joining $f \in F$ to $v \in V$ if and only if $v \in F$. Then $G$ is $m$-even.

Therefore in any colouring of $V$ with $m = n + 1$ colours, the number of multicoloured elementary simplices is $mc(G)$, which is even.

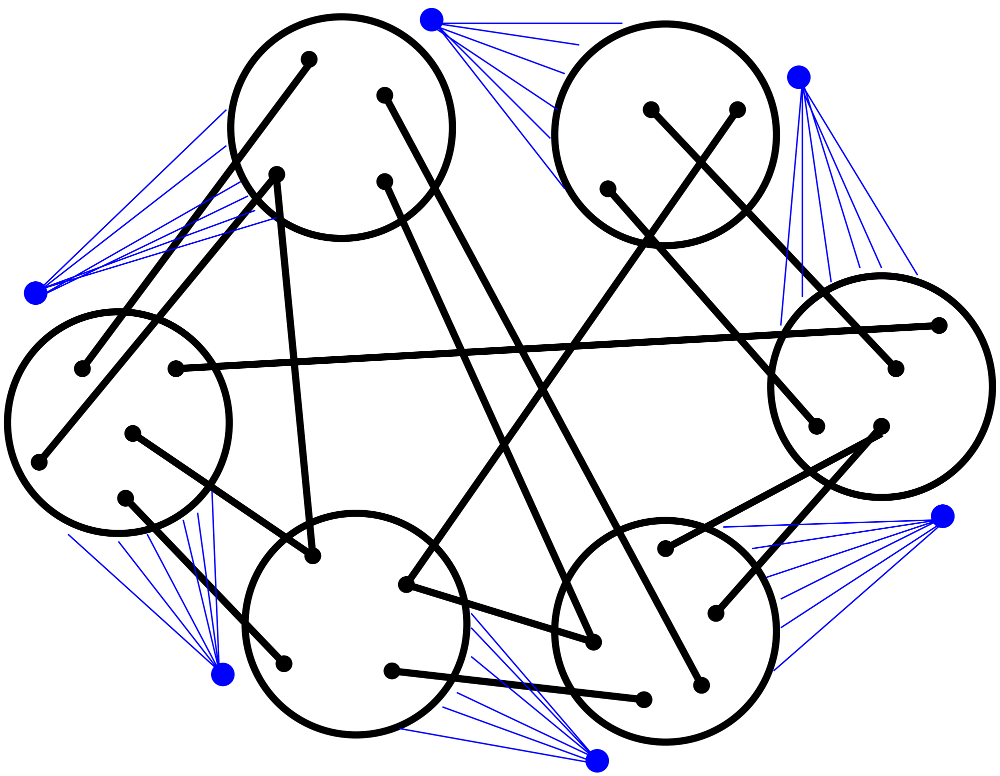
# An Application

Let $G$ be a graph with vertex partition $V_1, \ldots, V_m$. An independent transversal in $G$ is an independent set $\{v_1, \ldots, v_m\}$ in $G$ such that $v_i \in V_i$ for each $i$.




INDEPENDENT
TRANSVERSAL

Let $G$ be a graph with vertex partition $V_1, \ldots, V_m$. Define $G^+$ by adding a new vertex $x_i$ to each $V_i$ and all edges from $x_i$ to $V_{i+1}$.



THEOREM: Let $G$ be a graph with vertex partition $V_1, \dots, V_m$. Suppose there exists a set $\mathcal{C}$ of independent $m$-sets of vertices in $G^+$ such that

- every $(m-1)$-set is contained in an even number of elements of $\mathcal{C}$,
- $I^+ = \{x_1, \dots, x_m\} \in \mathcal{C}$.

Then $G$ has an independent transversal.

We call $\mathcal{C}$ a Special Independent Family in $G^+$.

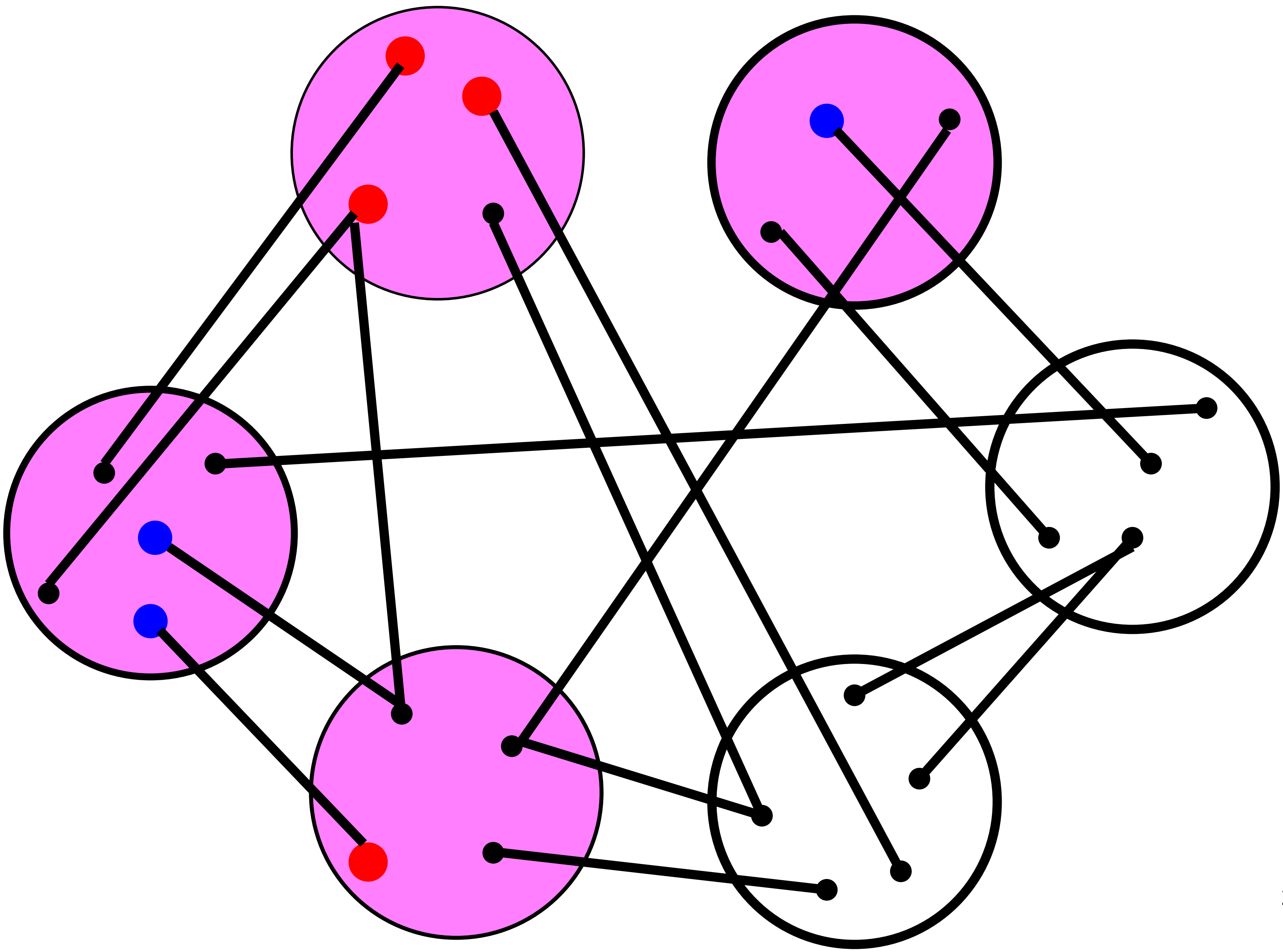
THEOREM: Let $G$ be a graph with vertex partition $V_1, \ldots, V_m$. Suppose $|V_i| \geq 2\Delta(G)$ for each $i$. Then $G^+$ has a SIF.

Follows from Aharoni-Chudnovsky-Kotlov, H

THEOREM: Let $G$ be a graph with vertex partition $V_1, \ldots, V_m$. Suppose that for every $S \subset \{1, \ldots, m\}$ there exists an independent set $I_S$ in $G_S = G[\cup_{s \in S} V_s]$ such that

every independent set $T$ in $G_S$ of size at most $|S| - 1$ can be extended by a vertex of $I_S$. Then $G^+$ has a SIF.

Follows from Aharoni-H



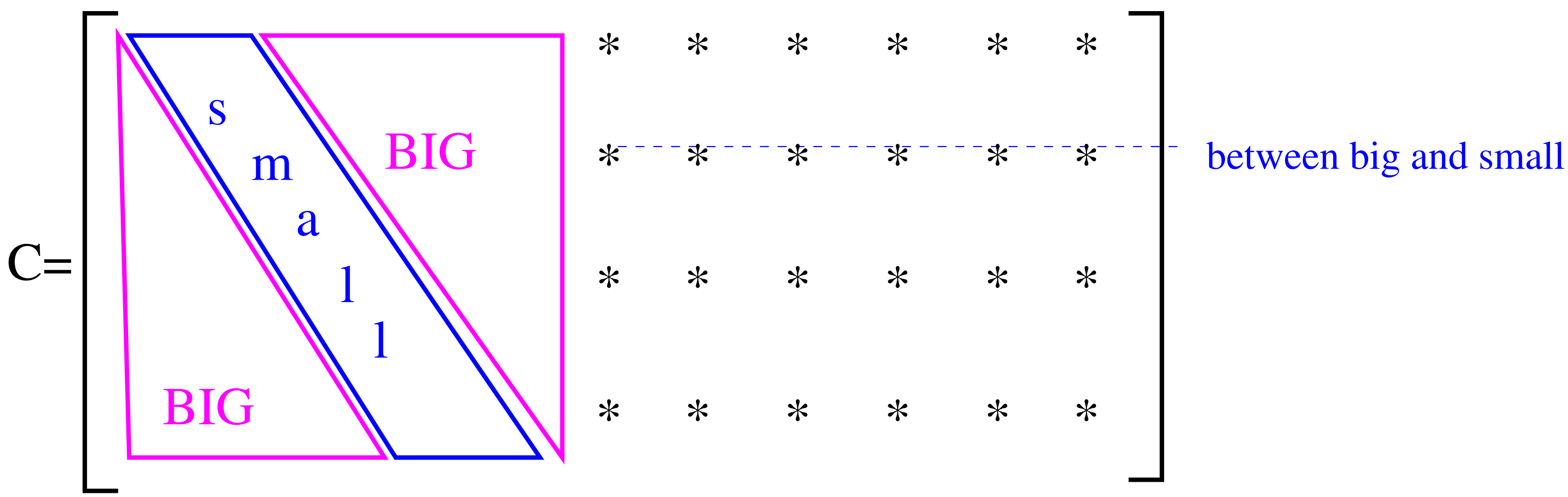
# Scarf's Lemma

Let $C$ be an $m \times t$ matrix where $m \leq t$. A set of columns $S$ is called dominating if

for every column $j$ there exists a row $i$ such that $a_{ij} \leq a_{is}$ for EVERY column $s \in S$.

$$\begin{bmatrix} 2 & 1 & 7 & 4 & 5 & 2 \\ 8 & 2 & 2 & 3 & 6 & 7 \\ 1 & 3 & 2 & 2 & 1 & 4 \\ 4 & 4 & 8 & 1 & 5 & 5 \end{bmatrix}$$

We are interested in dominating sets in matrices of this form:




where all entries in each row are distinct.

Note that the set consisting of first $m$ columns is NOT a dominating set of columns in $C$.

An example of a dominating set of columns in $C$: the first $m-1$ columns, together with the column indexed $k > m$ with the LARGEST entry in row $m$.

## Scarf's Lemma (simple version)

Let $C$ be an $m \times t$ matrix where $m < t$, whose columns are coloured with $m$ colours, satisfying the following properties:

- The first $m$ columns of $C$ are multicoloured,
- all entries in each row are distinct,
- $c_{ii} < c_{ik} < c_{ij}$ for each $k > m$ and $j \leq m$, $i \neq j$.

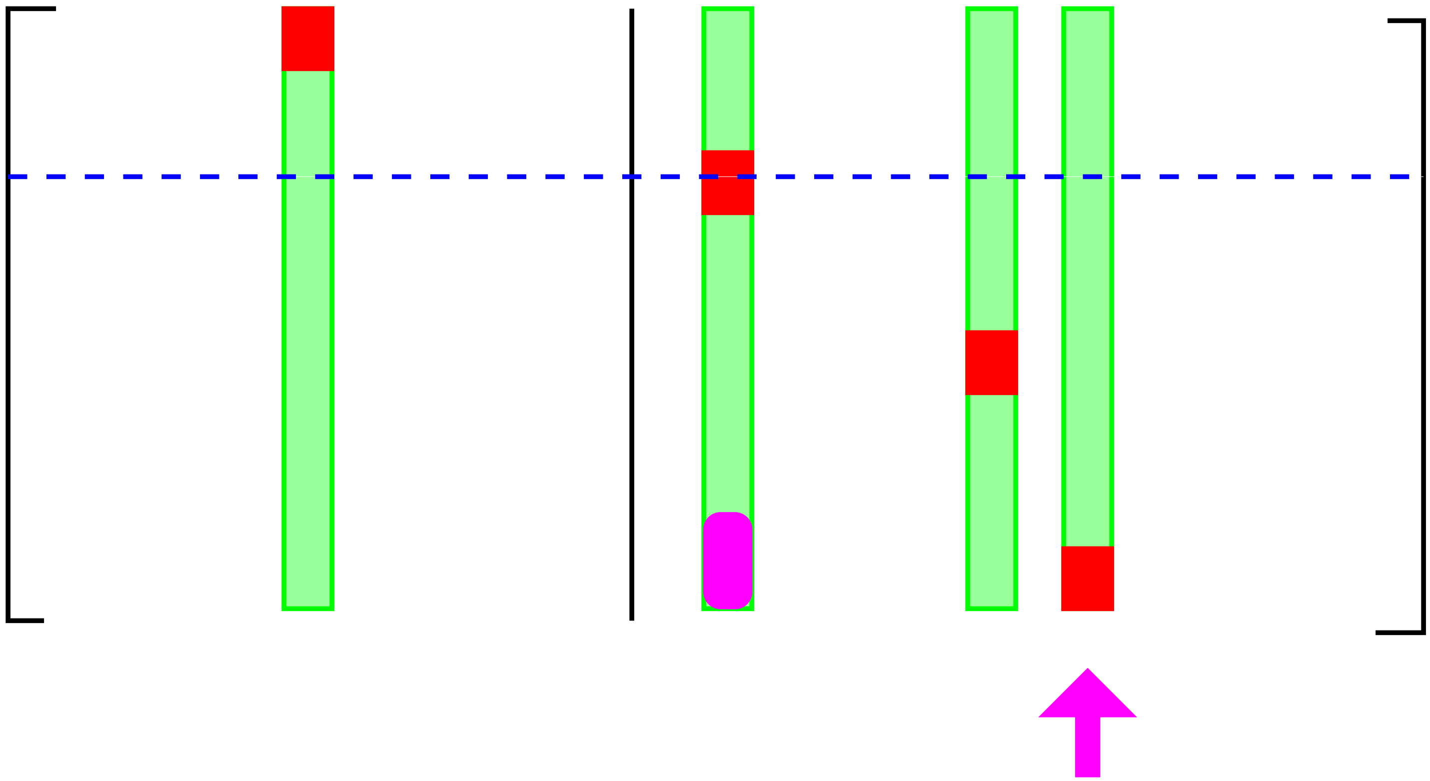
Then the number of multicoloured dominating $m$-sets of columns of $C$ is ODD. In particular there is at least one.

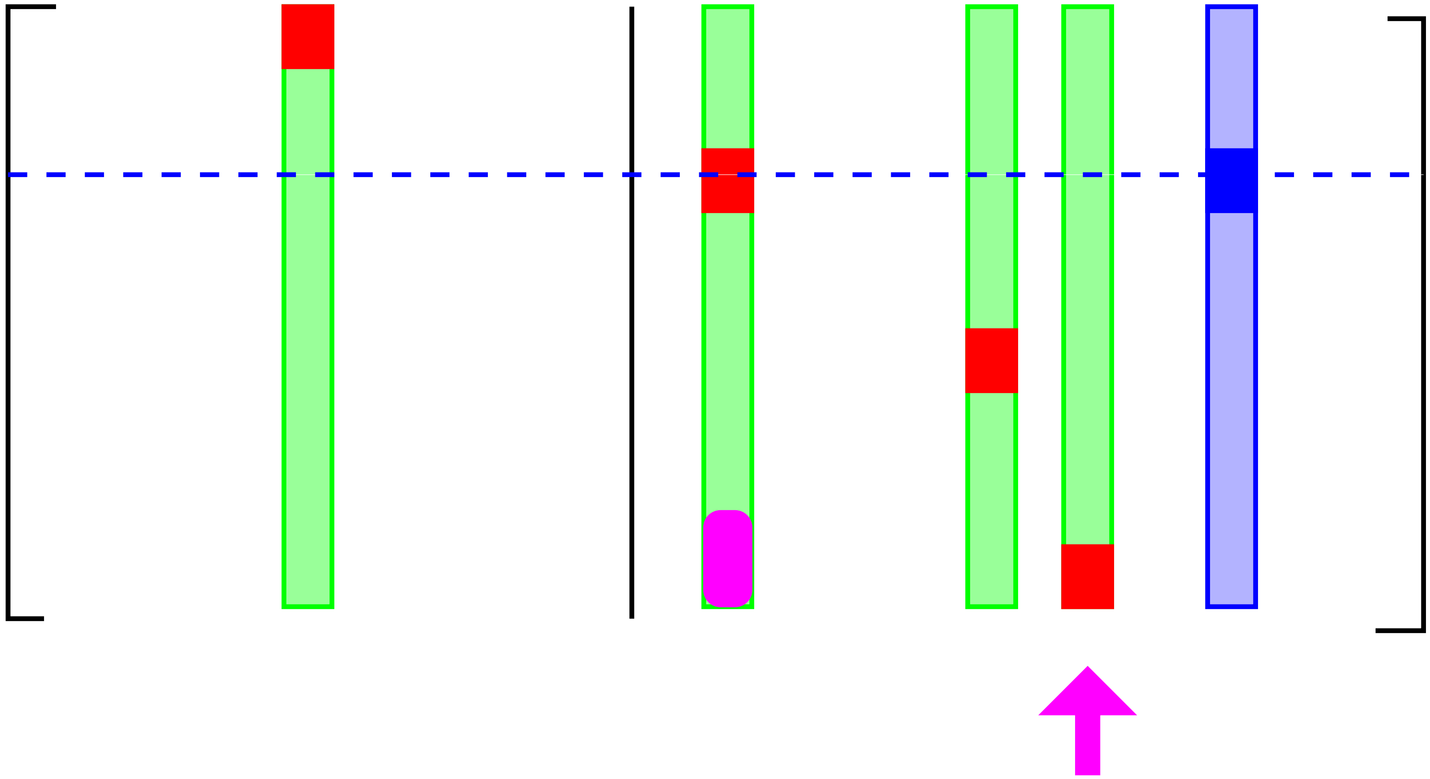
Define the set $F$ of $m$-sets of columns of $C$: the elements of $F$ are

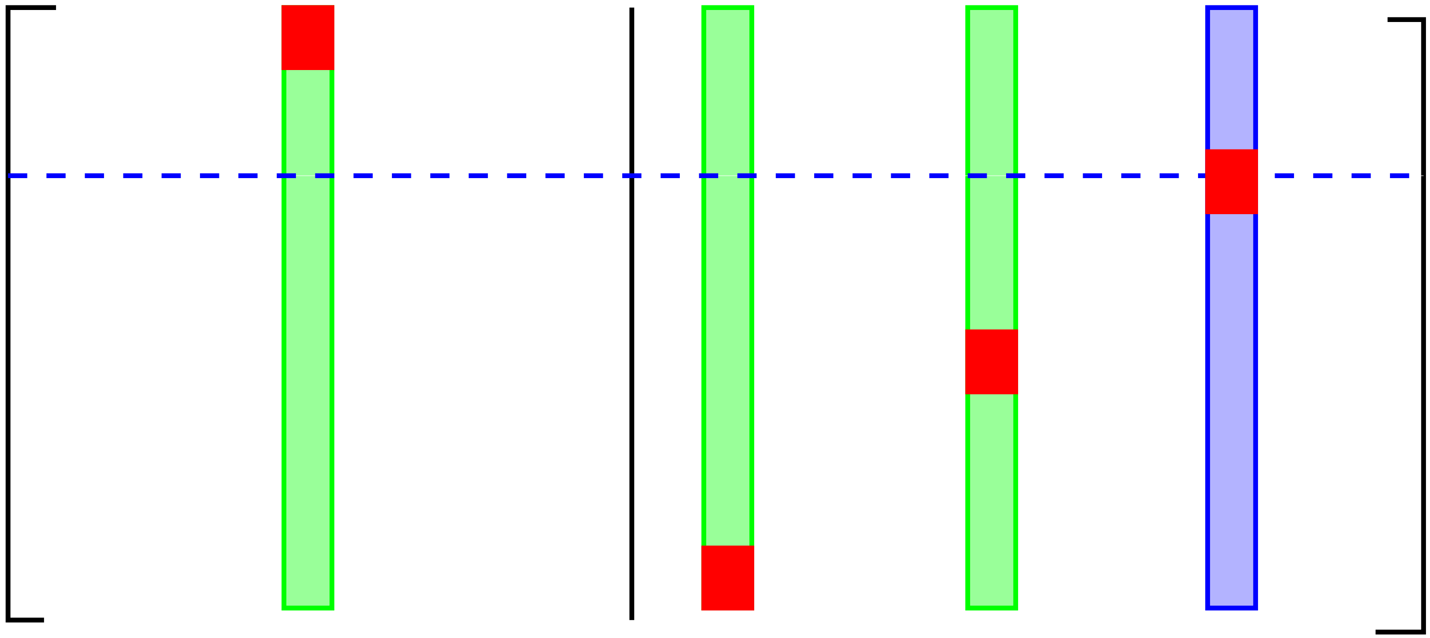
- the dominating $m$-sets of columns of $C$, together with
- the $m$-set consisting of the first $m$ columns.

Then (from the proof of Scarf's Lemma)

each $(m-1)$-set of columns of $C$ is either contained in two elements of $F$ or no elements of $F$.







## Scarf's Lemma (simple version)

Let $V$ be the set of columns of $C$. Form a bipartite graph $G$ with vertex sets $V$ and $F$ by joining $f \in F$ to $v \in V$ if and only if $v \in F$. Then $G$ is $m$-even.

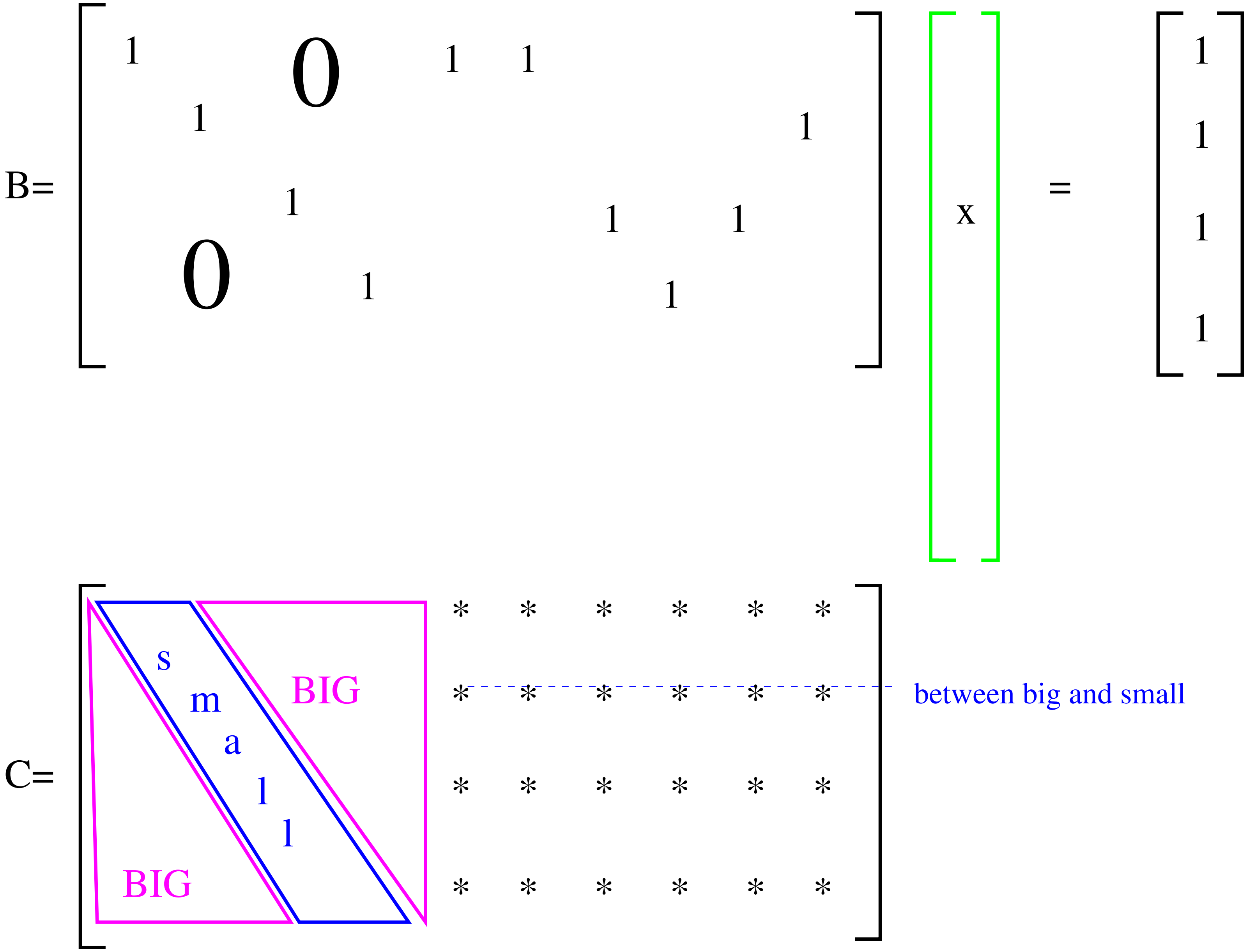
Therefore in any colouring of $V$ with $m = n + 1$ colours, the number of multicoloured $m$-sets in $F$ is $mc(G)$, which is even. Therefore since the set of the first $m$ columns is multicoloured but not dominating the number of multicoloured dominating $m$-sets is ODD. In particular, there is at least one.

# Scarf's Lemma (simple version)

Let $C$ be as before. Let $B$ be an $m \times t$ matrix and let $b \in R_+^m$ be the all-1 vector. Suppose $B$ satisfies

- The first $m$ columns of $B$ form an identity matrix,
- each column of $B$ contains exactly one 1 and all other entries are zero.

Then the number of 0-1 vectors $x$ that satisfy $Bx = b$ and whose support $supp(x)$ is dominating in $C$ is ODD. In particular, there is at least one.


B=
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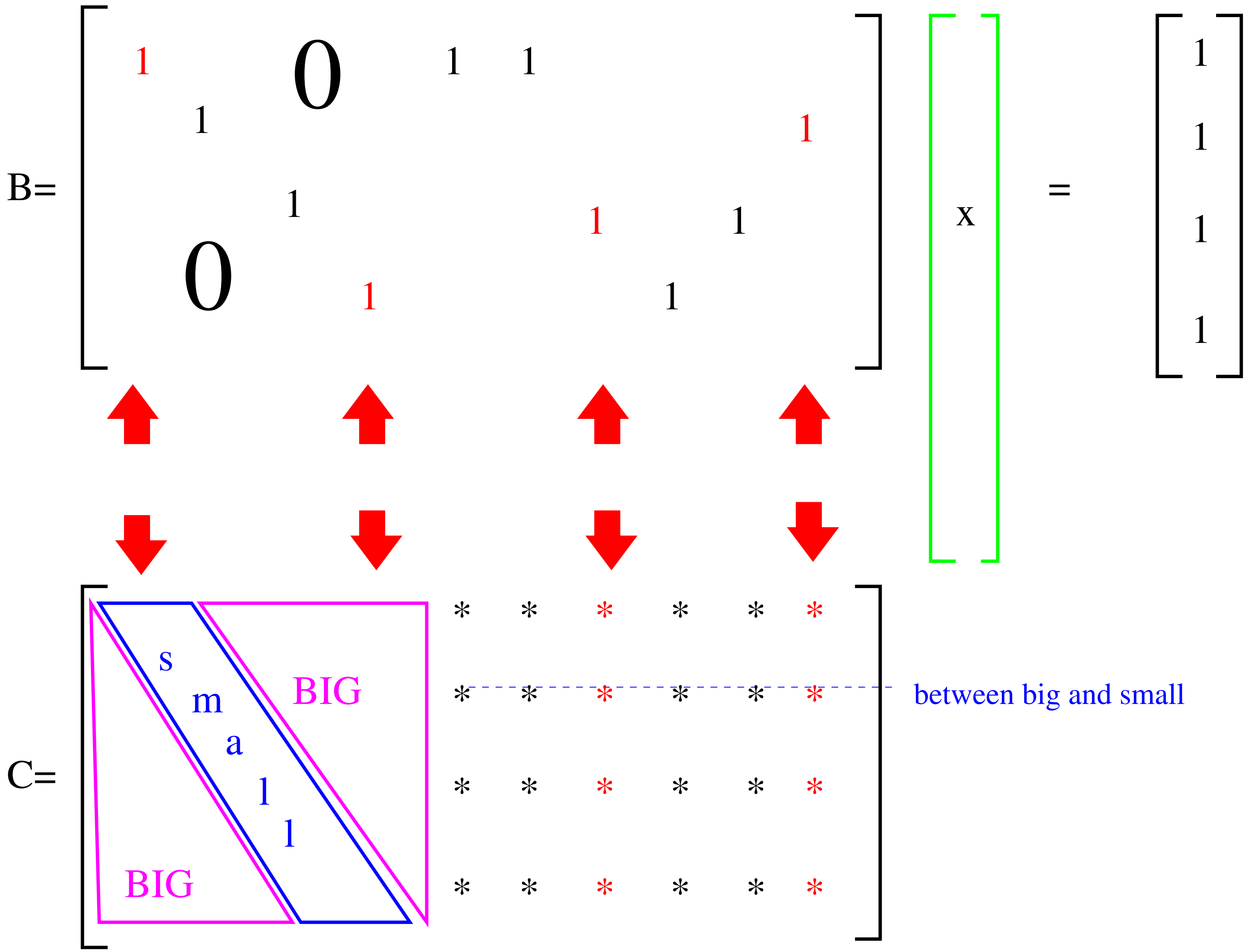
between big and small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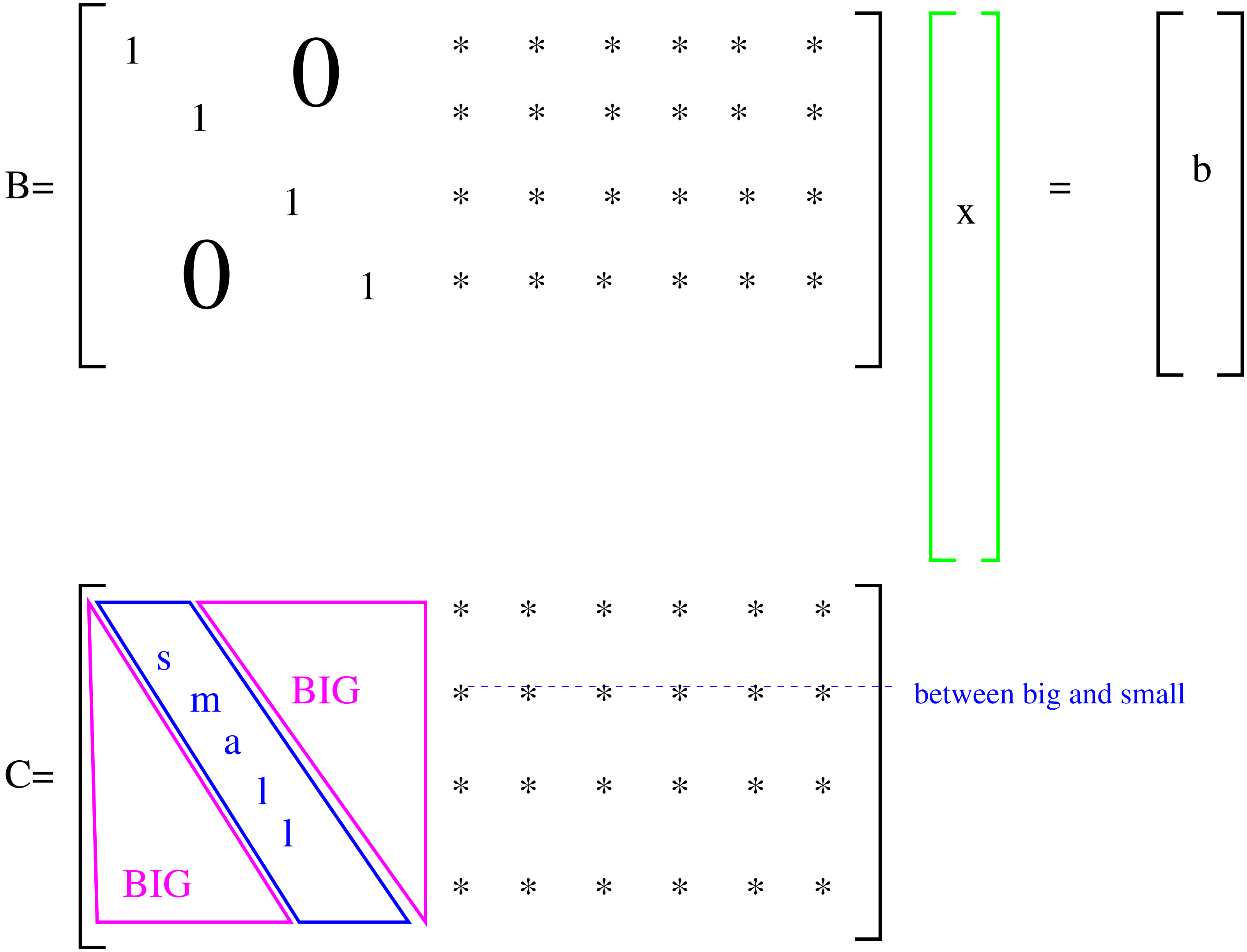
between big and small

## Scarf's Lemma (general version)

Let $B$ and $C$ be $m \times t$ matrices where $m < t$, and let $b \in R_+^m$ with the following properties (plus some suppressed conditions):

- The first $m$ columns of $B$ form an identity matrix,
- the set of non-negative solutions $x \in R_+^t$ to $Bx = b$ is bounded,
- $c_{ii} < c_{ik} < c_{ij}$ for each $k > m$ and $j \leq m$, $i \neq j$.

Then the number of non-negative solutions $x$ to $Bx = b$ whose support $supp(x)$ is dominating in $C$ is ODD. In particular, there is at least one.


B=
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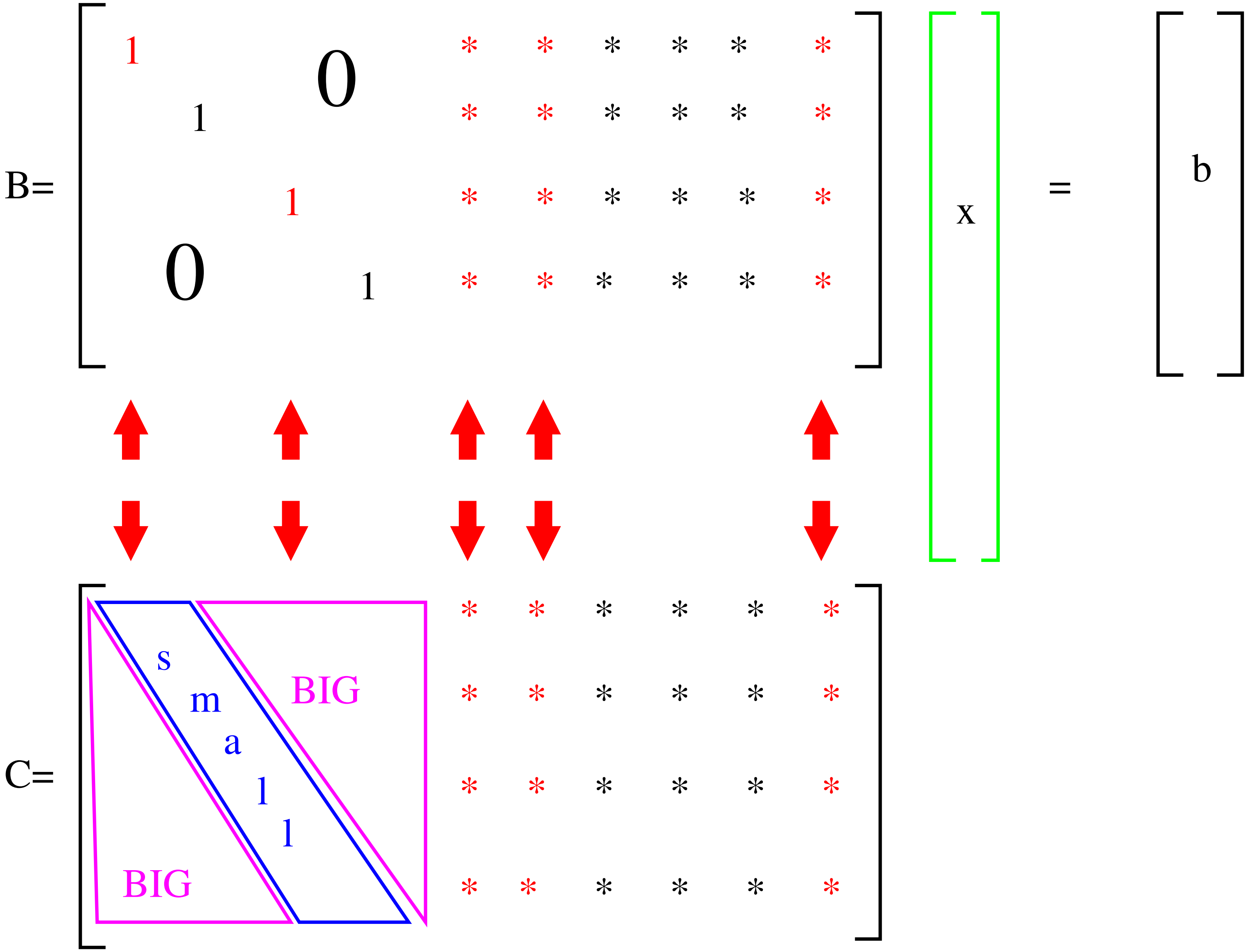
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Note that if $b$ has all positive entries then the set consisting of the first $m$ columns IS the support of a solution $x$ to $Bx = b$, but recall it is NOT a dominating set in $C$.

# Scarf's Lemma

Let $B$ and $C$ be matrices satisfying the conditions of Scarf's Lemma. Let $V$ denote the set of column indices of $B$ and $C$. We define two $m$-uniform hypergraphs on the vertex set $V$ as follows:

$\mathcal{B}$: those $m$-sets of columns that are the support of a solution $x$ to $Bx = b$. (Then every $(m-1)$-set of columns contains zero or two elements of $\mathcal{B}$.)

$\mathcal{C}$: those $m$-sets of columns that are dominating in $C$ TOGETHER WITH the single $m$-set consisting of the first $m$ columns.(Then every $(m-1)$-set of columns is contained in zero or two elements of $\mathcal{C}$.)

Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

## Sperner's Lemma for Restricted Triangulations

Let $m = (n + 1)$, and let an $m$-coloured restricted triangulation $T$ of an $n$-simplex be given. Let $V$ denote the set of points of $T$. We define two $m$-uniform hypergraphs on the vertex set $V$ as follows:

$\mathcal{B}$: those $m$-sets that are multicoloured (Then every $(m+1)$-set of points contains zero or two elements of $\mathcal{B}$.)

$\mathcal{C}$: those $m$-sets that are the vertices of an elementary simplex. (Then every $(m-1)$-set of columns is contained in zero or two elements of $\mathcal{C}$.)

Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

# A general theorem

THEOREM: Let $\mathcal{B}$ and $\mathcal{C}$ be $m$-uniform hypergraphs on the same vertex set $V$. Suppose

- each $(m+1)$-set contains an EVEN number of elements of $\mathcal{B}$.
- each $(m-1)$-set is contained in an EVEN number of elements of $\mathcal{C}$.

Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

PROOF: Same idea as proof of Scarf's Lemma or Sperner's Lemma. See e.g. Kuhn 1968, Aharoni and Fleiner 2003.

# Application: The Stable Paths problem

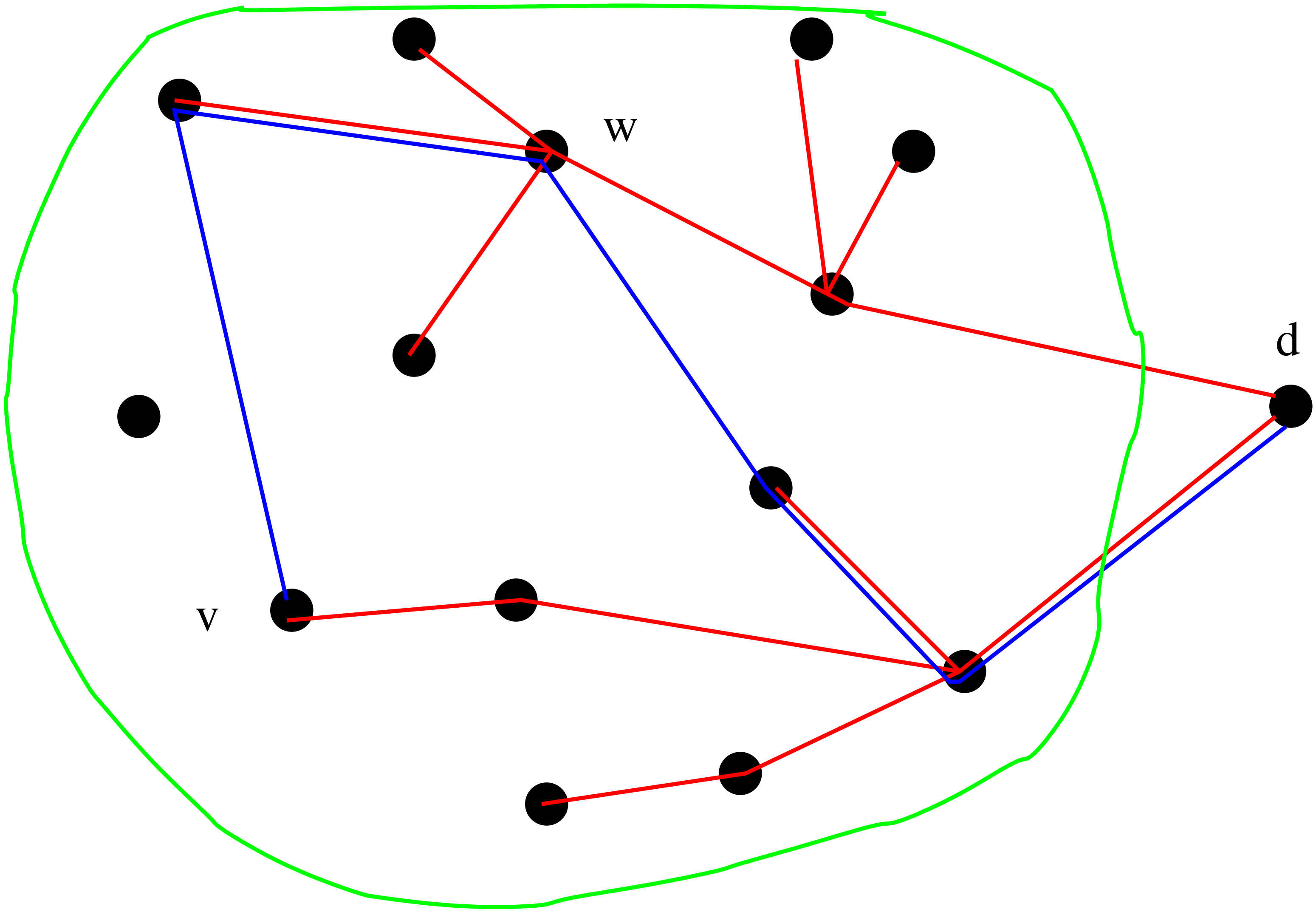
INSTANCE:

- A graph $G$ with a distinguished vertex $d$ (the destination),
- an ordered list $\pi(v)$ of paths from $v$ to $d$ for each vertex $v$ (the preference list of $v$).

SOLUTION: a tree $T$ in $G$, rooted at $d$, such that for every vertex $v$ and path $P \in \pi(v)$, either

- $v$ prefers its path in $T$ to $P$, or
- there is a PROPER final segment of $P$ that is not contained in $T$.

Motivation: internet routing protocols (Border Gateway Protocol)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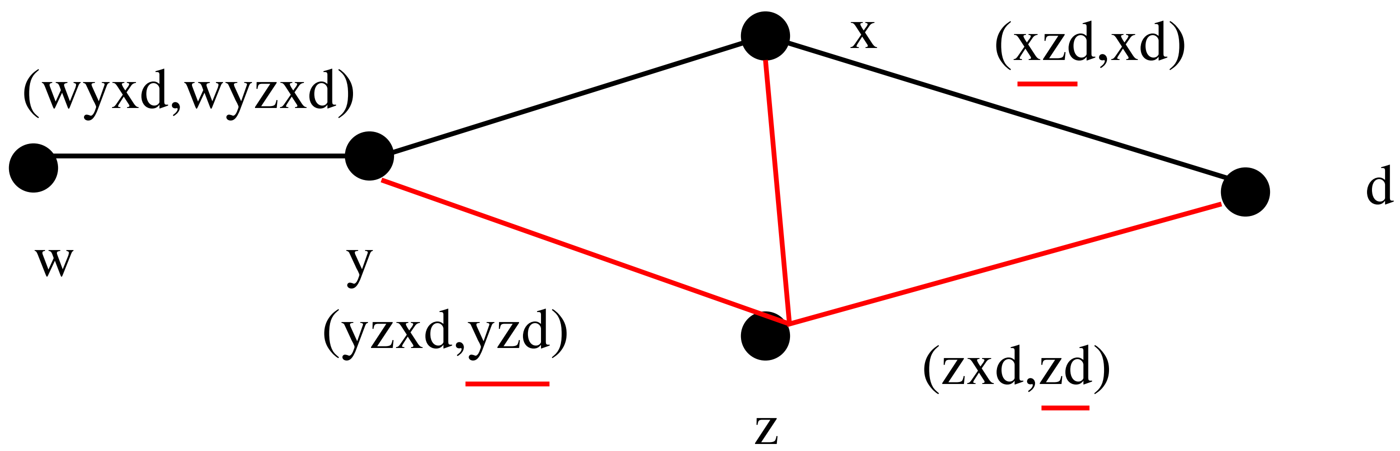

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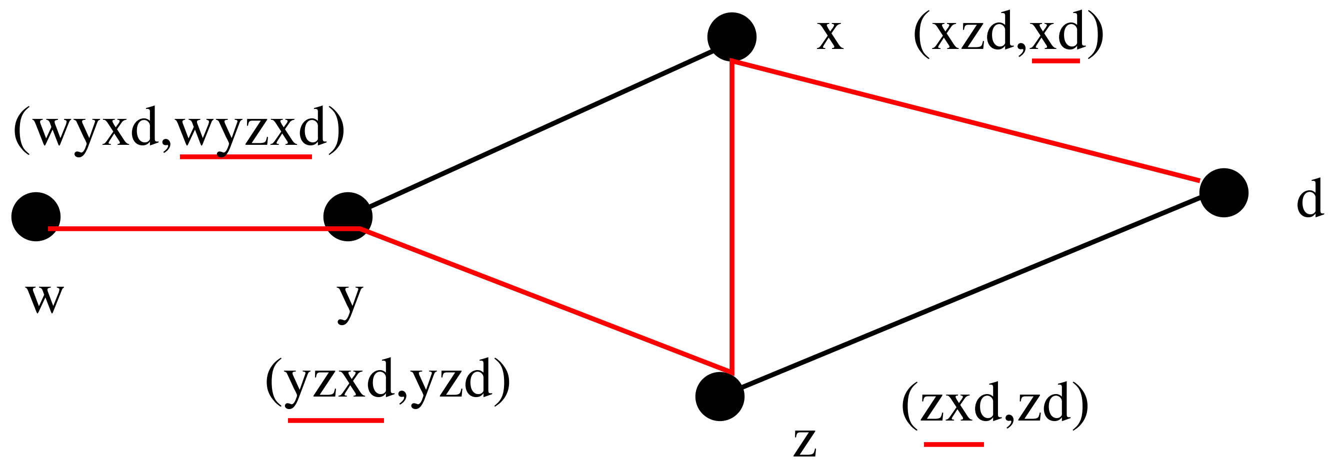
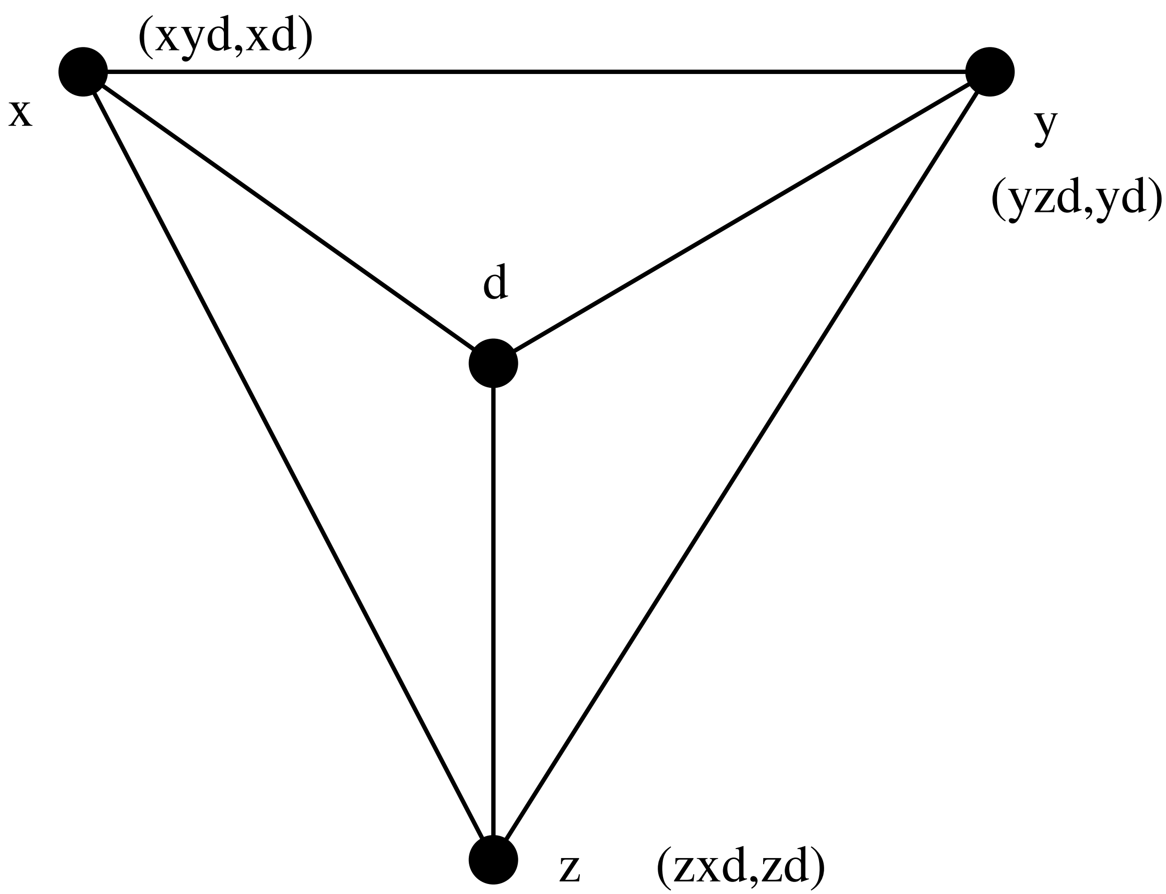
(wyxd,wyzxd)
d
w
y
(yzxd,yzd)
(zxd,zd)
z

x
(xzd,xd)
(wyxd,wyzxd)
d
w
y
(yzxd,yzd)
(zxd,zd)
z

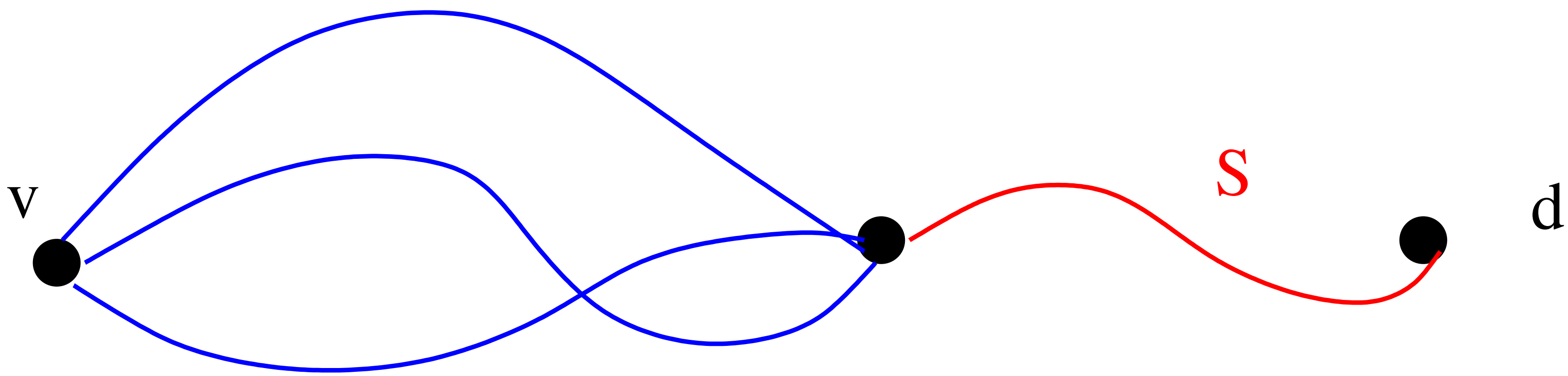
NOT EVERY instance of SPP has a solution:



# A fractional version

SOLUTION: A function that assigns a weight $w(P)$ to each path $P \in \cup_v \pi(v)$ such that

- for each $v$, $\sum_{P \in \pi(v)} w(P) \leq 1$,
- (tree condition) for each vertex $v$ and path $S$, $\sum_{P \in \pi(v,S)} w(P) \leq w(S)$, where $\pi(v, S)$ denotes the set of paths in $\pi(v)$ that end with the segment $S$,



(stability condition) for each $v$ and each $P \in \pi(v)$, either

- $\sum_{Q \in \pi(v)} w(Q) = 1$ AND $v$ prefers ALL its paths $Q \in \pi(v)$ for which $w(Q) > 0$ to $P$, or

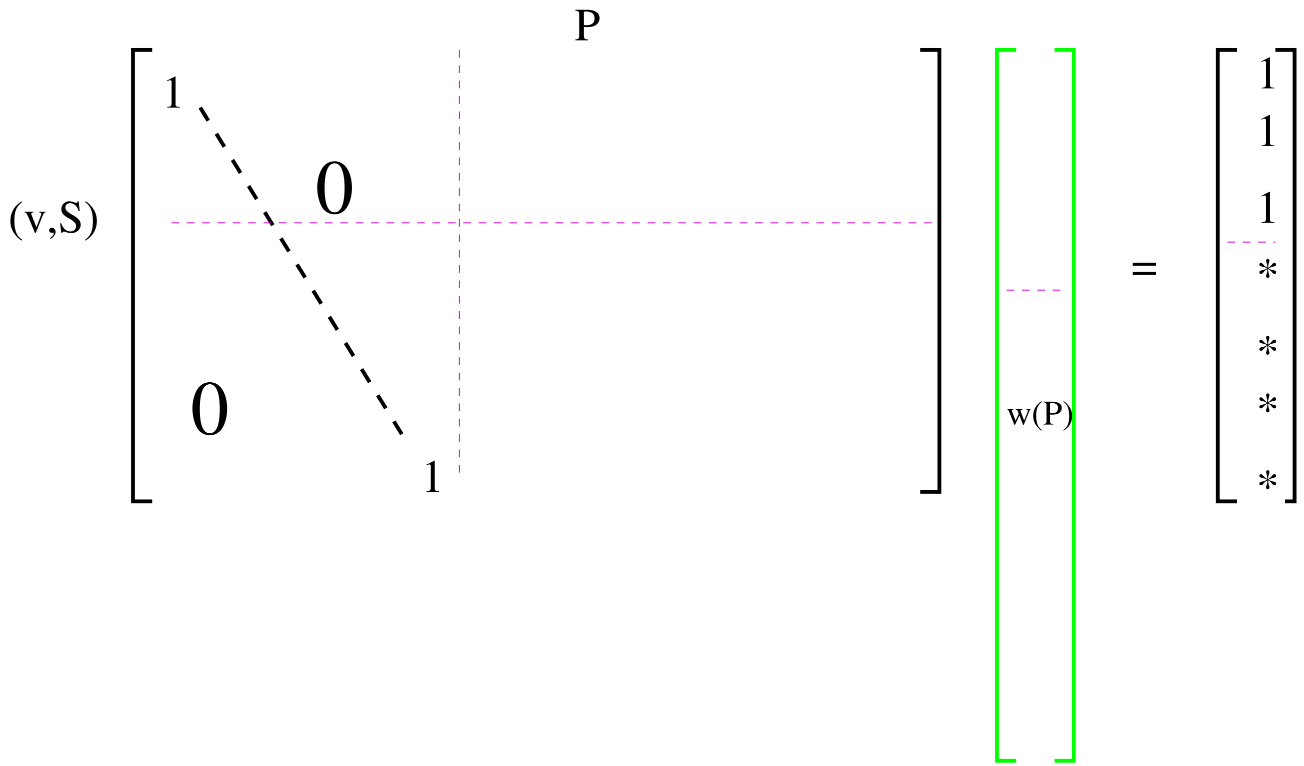
- there exists a PROPER final segment $S$ of $P$ such that $\sum_{Q \in \pi(v,S)} w(Q) = w(S)$ AND $v$ prefers ALL paths $Q \in \pi(v,S)$ for which $w(Q) > 0$ to $P$.

THEOREM (PH, Wilfong): Every instance of SPP has a fractional solution.

PROOF: Uses Scarf's Lemma.

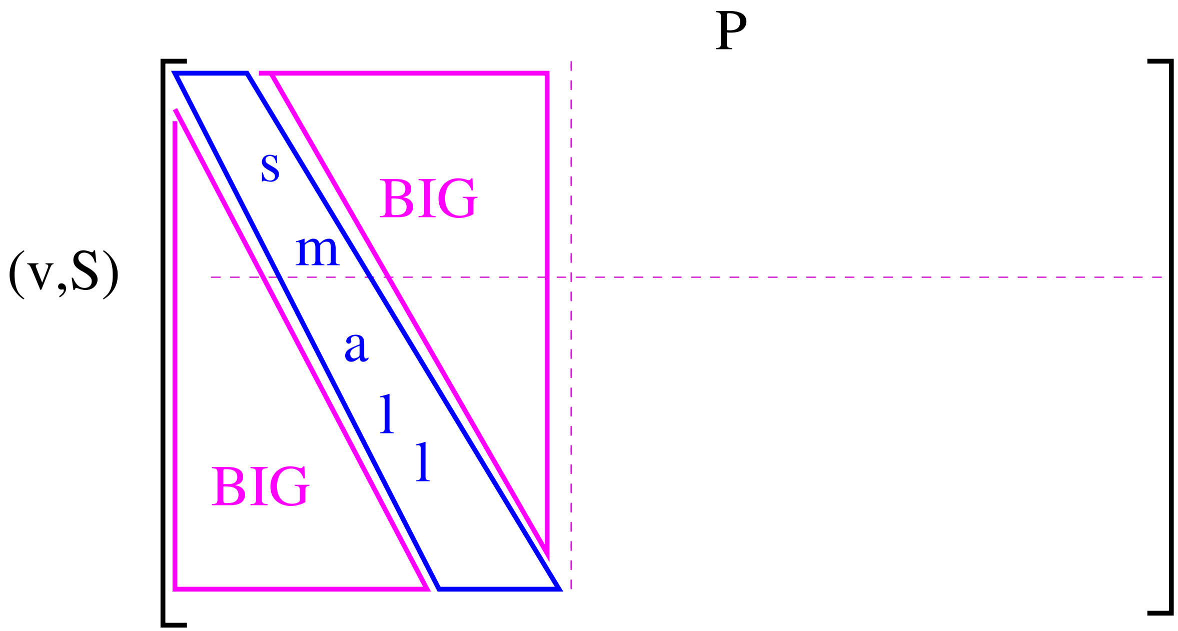
- The matrix $B$ encodes the tree condition.
- The matrix $C$ encodes the stability condition.
- The solution $x$ gives the weight function on paths that is the fractional solution to SPP.

# The matrix B




The $((v, S), P)$ entry is $-1$ if $P = S$, 1 if $P \in \pi(v, S)$, and 0 otherwise.

## The matrix C




The $((v,S),P)$ entry is the rank of $P$ in $\pi(v,S)$, if $P \in \pi(v,S)$, and $M$ otherwise, where $M$ is larger than any rank.

The solution $x$ from Scarf's Lemma gives a weight function $w$ on all paths in $\cup_v \pi(v)$.

The matrix $B$ ensures that the tree condition is satisfied.

The dominating property of $supp(x)$ ensures that the stability condition is satisfied.