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"Recall": events $\mathcal{A}, \mathcal{B}$ pos. corr. ($\mathcal{A} \uparrow \mathcal{B}$) if

$$\Pr(\mathcal{A} \mid \mathcal{B}) \geq \Pr(\mathcal{A}) \quad \Leftrightarrow \quad \Pr(\mathcal{A}\mathcal{B}) \geq \Pr(\mathcal{A}) \Pr(\mathcal{B}).$$

(neg. corr...)

$\mathcal{M}(\Omega) = \{$ prob. meas's on $\Omega \}$

$\Omega \to \{0,1\}^X = 2^X, \{\pm 1\}^X \ldots$

$\mu \in \mathcal{M}$ has:

(PC) pos. correlations if $\{\eta_e = 1\} \uparrow \{\eta_f = 1\}$ $\forall e, f$.

(NC) neg. corr's ...

(PA) pos. association if

$$\mathcal{A} \uparrow \mathcal{B} \quad \forall \text{ incr. } \mathcal{A}, \mathcal{B} \subseteq \{0,1\}^X$$

(NA) neg. assoc. ... NOT TODAY

e.g. $G=(V,E)$ (finite); $e,f \in E$

$\mathbb{F}$ unif (sp.) forest ...

Conj 1 $\{e \in \mathbb{F}\} \downarrow \{f \in \mathbb{F}\}$

(true for unif. sp. tree ...)

"motivation":

- asymptotic normality ...
- mixing ...

Mason's Conj* (72): $a_i = \Pr(|\mathbb{F}| = i)$:

1. Log-Concave $(a_i^2 \geq a_{i-1} a_{i+1})$

2. Ultra LC: $\left\{ \frac{a_i}{\binom{n}{i}} \right\}$ LC.

* actually for matroids ...

SW (75): Conj 1 false for mat's

generalize* Conj 1:

$w: E \longrightarrow \mathbb{R}^+ \rightsquigarrow \Pr(\mathbb{F} = F) \propto \prod_{e \in F} w_e$

Conj 1' $\{e \in \mathbb{F}\} \downarrow \{f \in \mathbb{F}\}$

external fields (c.f. Ising model):

$\mu \in \mathcal{M}(\{0,1\}^X)$, $w: X \longrightarrow \mathbb{R}^+$:

$w \circ \mu(\eta) \propto \mu(\eta) \prod_{e \in X} w_e^{\eta_e}$

μ is Rayleigh if $w \circ \mu$ NC $\forall w$

[Conj 1': unif. forest meas's are Rayleigh

Conj ("physicists", Pemantle, Wagner)

Rayleigh $\Rightarrow$ ULC.

* not really

BBL/KN*: false

KN: not even unimodal

$\mathbb{F}$ again: (Conj 1 $\Rightarrow$)

Conj 2 $\forall\ s, t, u, v \in V$

$$\{s - t\} \uparrow \{u - v\}$$

Conj 3: E fin. $\subseteq W = \underbrace{\text{v.s.}}_{\text{or matroid}}$; $u, v \in W$

$\mathbb{F}$ unif lin. ind subset of E

$$\Longrightarrow \{u \in \underbrace{\langle \mathbb{F} \rangle}_{\text{span}}\} \uparrow \{v \in \langle \mathbb{F} \rangle\}$$

* Borcea-Brändén-Liggett / K-Neiman

<u>RCM</u>*: $G=(V,E)$; $p=(\underbrace{p_e}_{\to [0,1]} : e\in E)$

$q \underline{\underline{> 0}}$

$\rightsquigarrow \varphi = \varphi_{p,q} \in \mathcal{M}(\{0,1\}^E)$:

$$\varphi(\eta) \propto \prod_e p_e^{\eta_e}(1-p_e)^{1-\eta_e} \cdot q^{\underbrace{k(\eta)}_{\#\text{cpts}}}$$

- $q=1$: percolation
- connection w Ising — wait
- $\lim\limits_{p_e=q\to 0} \varphi_{p,q}$ = unif. forest meas.

(other lim's: unif. sp. tree
unif. conn. (sp.) subgph

(AFAIK) all above conj's could be okay
for RCM's w $q \underline{\underline{\leq 1}}$

* random cluster $\left\{\begin{matrix}\text{model}\\ \text{measures}\end{matrix}\right\}$; Grimmett...

"Divide & Color": $G=(V,E)$, $\alpha \in (0,1)$

$\varphi \in \mathcal{M}(\{0,1\}^E)$ ("edge configs")

$\rightsquigarrow$ σ random $\in \underline{\{\pm 1\}^V}$:

① choose $\eta \in \{0,1\}^E$ acc. φ

② flip coin for each cpt $\begin{cases} \text{indept.} \\ \Pr(+1) = \alpha \end{cases}$

e.g. $\varphi = \varphi_{p,q}$ $\quad q \underline{\underline{=2}}$, $\underline{\underline{\alpha = 1/2}}$ $\rightarrow$

<u>Ising</u> <u>model</u>

$\beta^* > 0$: $\quad \mu = \mu_\beta \in \mathcal{M}(\{\pm 1\}^V)$:

$$\mu(\sigma) \propto \exp\Big[\beta \sum_{v \sim w} \sigma_v \sigma_w\Big]$$

PA by FKG

* or β_e ...

general D&C:

- $\sigma_v \uparrow \sigma_w$* (TRIV)
- PA? not always...
- suff. cond's ??

Thm (H^*'99) YES if $\varphi = \varphi_{p,q}$ w

$\boxed{\alpha q, (1-\alpha) q \geq 1}$ via FKG...

thm (H-Schramm '01) YES if

$\varphi = \varphi_{p,q}$ w $q = 1$ (i.e. perc)

thm (K-W^*'05) YES $\forall$ $\varphi_{p,q}$ w $q \geq 1$

Conj Also for $q < 1$ (really?)

– e.g. unif forest

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* $\{\sigma_v = 1\} \uparrow \{\sigma_w = 1\}$ *Häggström *Weininger

e.g. "Shlosman's Conj" [still Ising]

(early 90's? also C. Kenyon-Mossel-Peres)

$\langle \cdot \rangle = \mathbb{E}$; $v, w \in V(G) \longrightarrow$

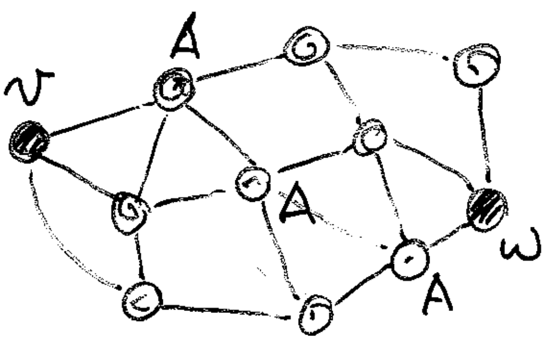
$I(v,w) = \langle \sigma_w \mid \sigma_v = 1 \rangle - \langle \sigma_w \mid \sigma_v = -1 \rangle$ $(\geq 0$

"influence"

$A \subseteq V$, $\tau \in \{\pm 1\}^A \longrightarrow$

$$I_\tau(v,w) = \langle \sigma_w \mid \sigma_v = 1, \sigma \equiv \tau \text{ on } A \rangle$$
$$- \langle \sigma_w \mid \sigma_v = -1, \sigma \equiv \tau \text{ on } A \rangle$$

e.g.



$I_\tau(v,w) = 0$

$(\forall \tau \in \{\pm 1\}^A)$

Conj $I(v,w) \geq I_\tau(v,w)$ $(\forall G, v, w, A, \tau)$