THREE PROBLEMS IN EXTREMAL SET THEORY

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X - finite set

$\mathcal{G}$ - collection of subsets of X (of size k)

$\mathcal{F}$ - specific configuration of sets (of size k)

How large can $|\mathcal{G}|$ be without containing a copy of $\mathcal{F}$?

Example (Sperner 1928)

$|X| = n$; $\mathcal{F}$ — A, B s.t. $A \subset B$

$$|\mathcal{G}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

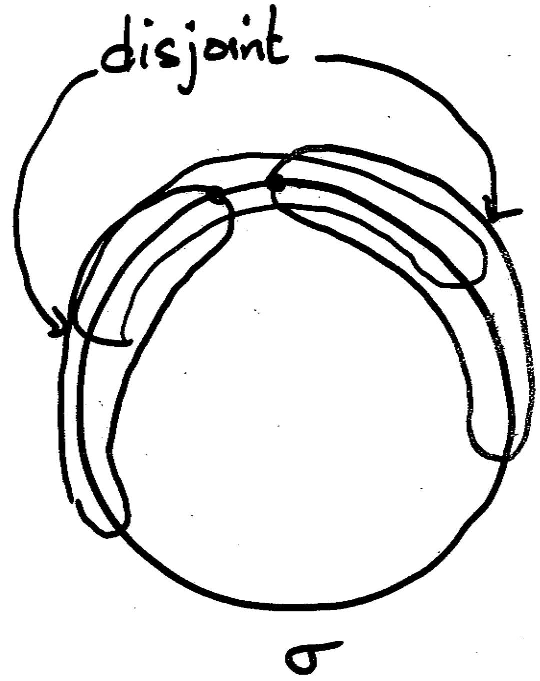
Intersecting families

Theorem (Erdös-Ko-Rado 1961)

Let $n \geq 2k$ and $\mathcal{G} \subset \binom{[n]}{k}$ be intersecting ($\mathcal{F} =$). Then

$$|\mathcal{G}| \leq \binom{n-1}{k-1}$$

Proof (Katona)



$$|\mathcal{G}| \leq \frac{\sum_{\sigma} |\mathcal{G}_\sigma|}{k!\,(n-k)!}$$

$$\leq \frac{(n-1)!\; k}{k!\;(n-k)!}$$

$$= \binom{n-1}{k-1}$$

Simplices

d-simplex — collection of $d+1$ sets $A_1, \dots, A_{d+1}$ s.t. $\bigcap A_i = \phi$ and $\bigcap_{i \neq j} A_i \neq \phi \ \forall j$

1-simplex — two disjoint edges

Conjecture (Chvátal 1972)
Let $2 \leq d \leq k$ and $n \geq \frac{(d+1)k}{d}$. Suppose $G \subset \binom{[n]}{k}$ contains no d-simplex.
Then $|G| \leq \binom{n-1}{k-1}$

$d = 1$ Erdös-Ko-Rado

$d = 2$ Conjecture of Erdös (1971)

Triangles

Erdös	1971	Made Conjecture
Chvátal	1974	$k=3$ and generalized
Frankl	1976	$\frac{3k}{2} \leq n \leq 2k$
Bermond-Frankl	1977	infinitely many n, k, $n < k^2$
Frankl	1981	$n > n_0(k) > c^k$
Frankl-Füredi	1987	Chvátals Conjecture for $n > c^k$
Csakany-Kahn	1999	New proof for $K=3$ using Homology
M-Verstraëte	2004	$n \geq \frac{3k}{2}$

Theorem (Frankl 1978
Wilson 1984
Ahlswede-Khachatrian 1997)

Let $n, k, t \geq 2$ and $\mathcal{G} \subset \binom{[n]}{k}$ be a t-intersecting family. Then

$$|\mathcal{G}| \leq M(n, k, t), \text{ where}$$

$$M(n,k,t) = \begin{cases} \binom{n-t}{k-t} & n \geq (t+1)(k-t+1) \\ \max_i |\mathcal{G}_i| & n \leq (t+1)(k-t+1) \end{cases}$$

$$\mathcal{G}_i = \left\{ S \in \binom{[n]}{k} : |S \cap [t+2i]| \geq t+i \right\}$$

Theorem (Frankl - Füredi 1987)

Chvátals conjecture holds for $n > n_0(k,d)$

What about $n < n_0(k,d)$?

Theorem (Keevash - M 2007^+)

For all $d \geq 2$ and $C > 2$ there are integers T, K_0 s.t. if $k > K_0$

$$2k + T < n < Ck$$

and $G \subset \binom{[n]}{k}$ contains no d-simplex,

then

$$|G| < \binom{n-1}{k-1}$$

unless G is a star

Problem (Erdös)

Let $\mathcal{G} \subset 2^{[n]}$ be triangle-free.

Then $|\mathcal{G}| \leq 2^{n-1} + n-1$.

Many proofs: Milner, Lossers, M-Verstraëte

Theorem (Keevash-M 2007)

Let $d \geq 2$ and $n > n_0(d)$. Suppose that $\mathcal{G} \subset 2^{[n]}$ contains no d-simplex. Then

$$|\mathcal{G}| \leq 2^{n-1} + \sum_{i=1}^{d-1} \binom{n-1}{i}$$

Clusters

Problem (Katona)

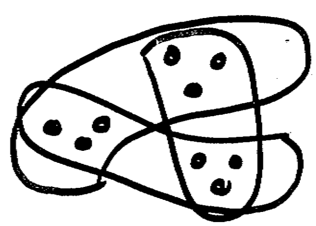
Let $\mathcal{G} \subset \binom{[n]}{k}$ s.t. for every

$A, B, C \in \mathcal{G}$ with $|A \cup B \cup C| \le 2k$,

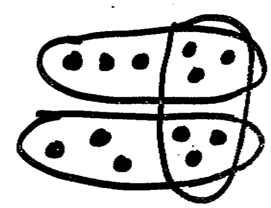
$$A \cap B \cap C \neq \emptyset.$$

Then $|\mathcal{G}| \le$??

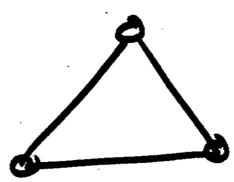
Forbidden

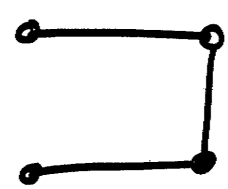


$k=6$



$k=2$ (Graphs)





Theorem (Frankl-Füredi 1983)

In Katona's problem, $|\mathcal{G}| \leq \binom{n-1}{k-1}$

as long as $n > k^2 + 3k$ (and this is sharp)

Conjecture (Frankl-Füredi 1983)

Result holds for ALL $n \geq \frac{3k}{2}$

d-cluster — collection of $d+1$

sets $A_1, \ldots, A_{d+1}$ s.t.

$$\left| \bigcup_{i=1}^{d+1} A_i \right| \leq 2k \quad \text{and}$$

$$\bigcap_{i=1}^{d+1} A_i = \phi$$

1-cluster — two disjoint edges

Conjecture (M 2005)

Let $1 \leq d \leq k$, $n \geq \frac{(d+1)k}{d}$ and $\mathcal{G} \subset \binom{[n]}{k}$ contain no d-cluster. Then

$$|\mathcal{G}| \leq \binom{n-1}{k-1}$$

- $d = 1$ (Erdös-Ko-Rado)
- $d = 2$ (M-2005, settles F-F conjecture)
- $d = k$ (follows from Chvátals result on simplices)

- $d = 3$ and $n > n_0(d,k)$ (M- 2007)
- $1 \leq d \leq k$ and $n > n_0(d,k)$ (M-Ramadurai, Füredi-Ozkahya 2007$^+$)

What about when n is smaller?

Theorem (Keevash-M 2007+)

For all $C > 2$ and $d \geq 2$, there exist integers T and k_0 s.t. if $k > k_0$

$$2k + T < n < Ck$$

and $G \subset \binom{[n]}{k}$ contains no d-cluster, then $|G| < \binom{n-1}{k-1}$ unless G is a star.

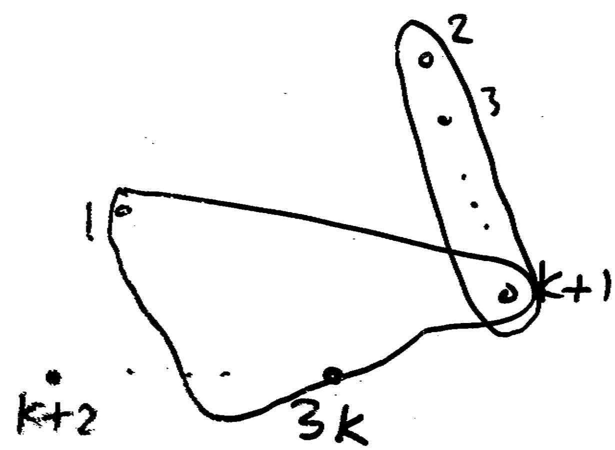
Result is false for $d = 1$

Counterexample

$n = 3k$

$$|G| \approx \binom{n-1}{k-1} - \binom{\frac{2n}{3} - 1}{k-1}$$

$$\sim \binom{n-1}{k-1}$$



Theorem (Keevash - M 2007[+])

For all $d \geq 2$, $C > 2$, there exists $\delta > 0$ and integers T and k_0 s.t. the following holds for $k > k_0$:

Suppose that

$$2k + T \leq n \leq Ck$$

and $G \subset \binom{[n]}{k}$ satisfies

$$|G| > \left(1 - \delta\left(1 - \frac{2k}{n}\right)\right)\binom{n-1}{k-1}.$$

If G contains no d-cluster, then G is a star. In particular,

$$|G| \leq \binom{n-1}{k-1}$$

Result is FALSE for $d = 1$.

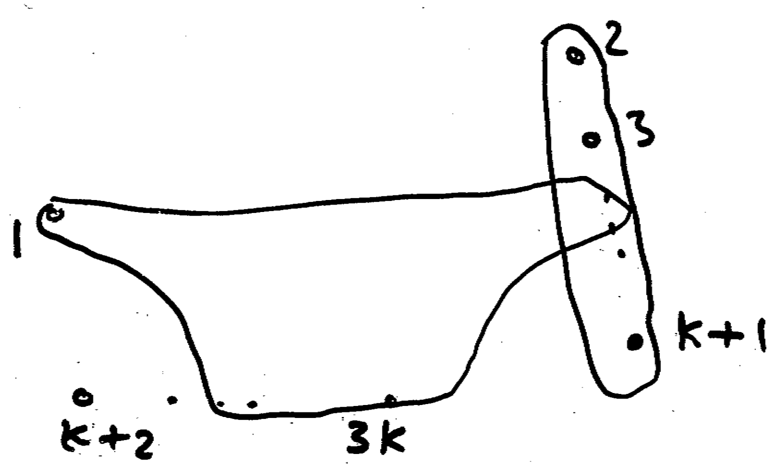
<u>Counterexample</u>:

$$n = 3k$$

$$|G| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$$

$$> \left(1 - \frac{\delta}{3}\right)\binom{n-1}{k-1}$$

- G is intersecting
- G is not a star



Intersecting families - Structure

Theorem (Keevash. M 2007$^+$)

For all $C > 2$, $\varepsilon > 0$ there exist $\delta > 0$ and integers T and k_0 s.t. the following holds for $k > k_0$:

Suppose that

$$2k + T \leq n \leq Ck$$

and $\mathcal{G} \subset \binom{[n]}{k}$ satisfies

$$|\mathcal{G}| > \left(1 - \delta\left(1 - \frac{2k}{n}\right)\right)\binom{n-1}{k-1}.$$

If $\mathcal{G}$ is intersecting, then there is $x \in [n]$ s.t. all but at most $\varepsilon\binom{n-1}{k-1}$ sets of $\mathcal{G}$ contain x.

Supersedes results of Friedgut proved via Fourier analysis

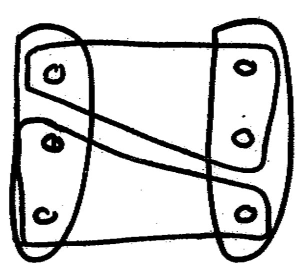
Two - Regular Subsystems

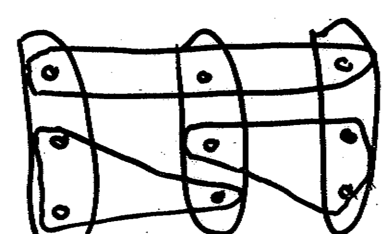
A 2-regular subsystem is a collection of sets that cover each point in their union exactly twice

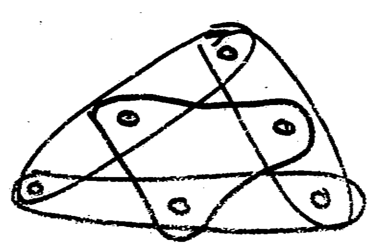
<u>Ex</u> For graphs

2-regular subsystem = cycle

<u>k=3</u>





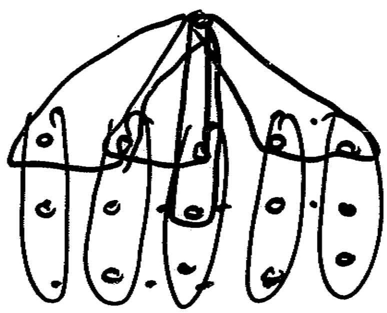


Note the difference

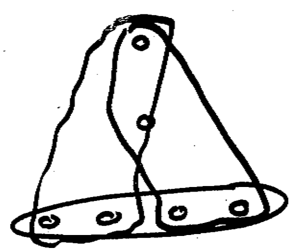
Constructions

$k=2$ Any tree — size $n-1$

$k=3$ Star + matching — size $\binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor$



$k=4$ Star — $\binom{n-1}{3}$



Conjecture (M-Verstraëte)

Fix $k \geq 2$. There exists $n = n_0(k)$ s.t. if $n > n_0(k)$ and $G \subset \binom{[n]}{k}$ contains no 2-regular subsystem then

$$|G| \leq \begin{cases} \binom{n-1}{k-1} & k \text{ even} \\ \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor & k \text{ odd} \end{cases}$$

Theorem (M - Verstraëte 2007^{+})

Fix $k \geq 2$. Then there exists $n_0(k)$ s.t. the following holds for $n > n_0(k)$: If $G \subset \binom{[n]}{k}$ contains no 2-regular subsystem, then

$$|G| \leq \begin{cases} \binom{n-1}{k-1} & k \text{ even} \\ \binom{n-1}{k-1} + n^{k-2} & k \text{ odd} \end{cases}$$

If $k \geq 4$ is even and equality holds above, then G is a star.

Methods

- Ahlswede-Khachatrian Thm
- Cross intersection Thm (Frankl)
- Erdös-Ko-Rado for paths (M-Verstraëte)
- Large Deviation Chernoff bounds
- Katona circle method
- Compressions
- Stability Approach !!

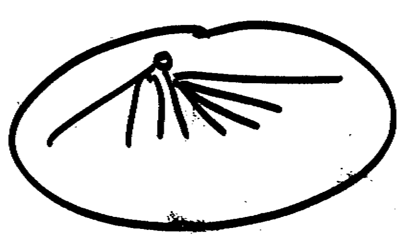
Stability

Introduced by Erdös-Simonovits in 1967 to solve problems in extremal graph theory

<u>Idea</u> If $F \not\subset G$ and $|G|$ is close to maximium, then the structure of G is well-understood

<u>Simple Example</u>

Suppose G is a graph with no pair of disjoint edges and no triangle and $|G| \sim n$. Then $G \sim$ star



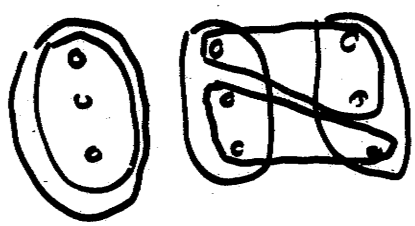
Counting Matchings

Observation (Lovász)

Let $G \subset \binom{[n]}{k}$ and $\mathcal{M}_1, \mathcal{M}_2$ be matchings in G such that

$$\bigcup_{S \in \mathcal{M}_1} S = \bigcup_{T \in \mathcal{M}_2} T .$$

Then $\mathcal{M}_1 \triangle \mathcal{M}_2$ is 2-regular



If # of matchings with s edges is greater than $\binom{n}{ks}$, then we have a 2-regular subgraph.

Lemma

Let $G \subset \binom{[n]}{3}$ contain no 2-regular subsystem. Then

$$|G| < 6\,\Delta^{2/3} \cdot n,$$

where $\Delta = \Delta(G)$

Proof (Sketch)

Count matchings of size $m = \dfrac{|G|}{3\Delta}$.

The number of these matchings is more than $\binom{n}{3m}$

Corollary

Every Steiner triple system contains a 2-regular subsystem

Speculation

- Stability results play an important role in solving extremal problems. The next few years should see more successes of this approach

- When stability is not an option, algebraic methods should apply, e.g.,
  - Li-Li proof of Grahams conj and Turán's Theorem (1981)
  - Razborovs work on counting triangles

HAPPY BIRTHDAY

TOM !!