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"SUBHYPERGRAPH COUNTS IN EXTREMAL & RANDOM HYPERGRAPHS AND THE FRACTIONAL q-INDEPENDENCE"

with J. Polcyn &
A. Dudek

$N(F,G) = \#$ copies of $G \subseteq F$

$G(n,p)$, $0 < p = p(n) < 1$

$X_G = X_G(n,p) = N(G(n,p), G)$

$$\mu_G = \mathbb{E} X_G = \underbrace{\binom{n}{v_G} \frac{v_G!}{\mathrm{aut}(G)}}_{N(K_n, G)} p^{e_G} = \Theta\Big(\underbrace{n^{v_G} p^{e_G}}_{\Psi_G}\Big)$$

<u>Threshold</u> (Bollobás 81)

$$n^{-1/m_G}, \quad m_G = \max_{H \subseteq G} \frac{e_H}{v_H}$$

<u>The lower tail</u> (Janson 90)

$$P(X_G \le (1-\varepsilon)\mu_G) = \exp\left\{-\Theta_\varepsilon\left(\min_{H \subseteq G} \Psi_H\right)\right\}$$

EXAMPLE $G = K_3 =$ 

$m_G = 1$

$$P(X_G = 0) \to \begin{cases} 1 & \text{if } p \ll \frac{1}{n} \\ 0 & \text{if } p \gg \frac{1}{n} \end{cases}$$

$$\min_{H \subseteq G} \Psi_H = \begin{cases} n^3 p^3 & \text{if } p \leq n^{-\frac{1}{2}} \\ n^2 p & \text{if } p \geq n^{-\frac{1}{2}} \end{cases}$$

$$P(X_G \leq (1-\varepsilon)\mu_G) = \begin{cases} e^{-\Theta_\varepsilon(n^3 p^3)}, & p \leq n^{-\frac{1}{2}} \\ e^{-\Theta_\varepsilon(n^2 p)}, & p \geq n^{-\frac{1}{2}} \end{cases}$$

Upper tail?

$$P(X_G \geq (1+\varepsilon)\mu_G)$$

Upper tail?

Rödl-R., Janson-R., Vu

— partial results

Janson, Oleszkiewicz, R. 2004:

$$P(X_G \geq (1+\varepsilon)\mu_G) \leq \frac{\mathbb{E}(X_G^m)}{(1+\varepsilon)^m \mu_G^m}$$

by Markov

<u>GOAL</u>: Find largest m:

$$\mathbb{E}(X_G^m) \leq \left(1+\frac{\varepsilon}{2}\right)^m \mu_G^m$$

Let $G_1, G_2, \ldots, G_{N(K_n,G)}$ be all copies of $G \subseteq K_n$

$$\mathbb{E}(X_G^m) = \sum_{i_1,\dots,i_m} \mathbb{E}(I_{G_{i_1}} \cdots I_{G_{i_m}}) =$$

$$= \sum p^{e(G_{i_1} \cup \dots \cup G_{i_m})} =$$

$$= \sum_{i_1,\dots,i_{m-1}} p^{e(F)} \sum_{i_m} p^{e_G - e(F \cap G_{i_m})} \leq$$

where $F = G_{i_1} \cup \dots \cup G_{i_{m-1}}$

$$\leq \sum p^{e(F)} \left\{ \mu_G + \sum_{\substack{H \subseteq G \\ e_H > 0}} \sum_{i : G_i \cap F \cong H} p^{e_G - e_H} \right\}$$

$$\leq \sum p^{e(F)} \left\{ \mu_G + \sum_H N(F,H)\, n^{v_G - v_H} p^{e_G - e_H} \right\}$$

$$N(n,m,H) = \max\{N(F,H) : v_F \leq n,\ e_F \leq m\}$$

In our case : $v_F \leq n,\ e_F \leq (m-1)e_G$

$$\Rightarrow N(F,H) \leq N(n,(m-1)e_G,H) = \Theta(N(n,m,H))$$

So,

$$\mathbb{E}(X_G^m) \leq \mathbb{E}(X_G^{m-1})\mu_G\left\{1+\sum_H \Theta\left(\frac{N(n,m,H)}{\Psi_H}\right)\right\}$$

$\Downarrow$ induction on m

$$\mathbb{E}(X_G^m) \leq \mu_G^m\left\{1+\sum_H \Theta\left(\frac{N(n,m,H)}{\Psi_H}\right)\right\}^{m-1}$$

$$M_G = M_G(n,p) =$$
$$= \max\left\{m \leq \binom{n}{2} : \forall H \subseteq G\ N(n,m,H) \leq \Psi_H\right\}$$

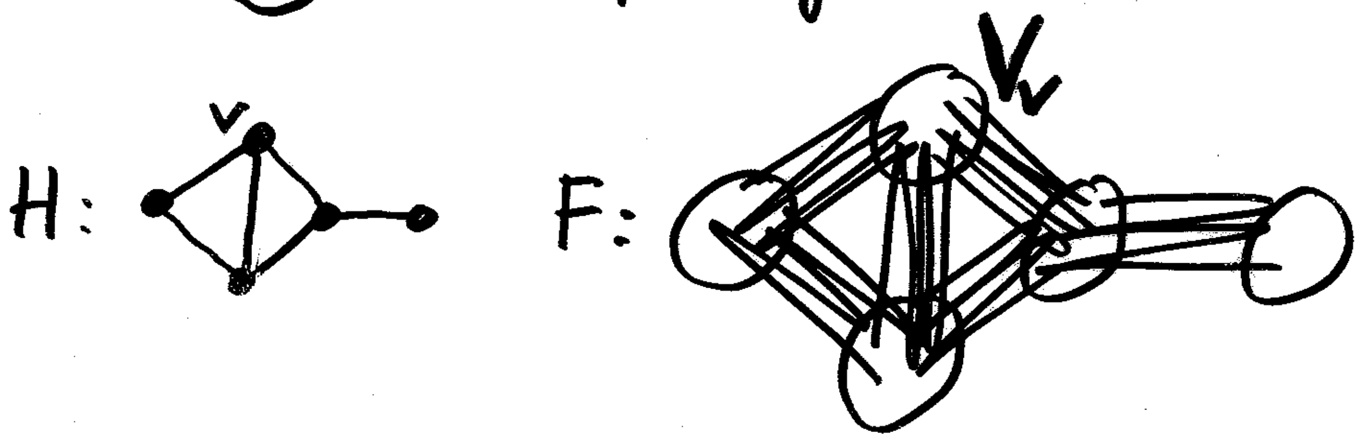
$$P(X_G \geq (1+\varepsilon)\mu_G) \leq \exp\{-\Theta_\varepsilon(M_G)\}$$

Also $\geq \exp\{-\Theta_\varepsilon(M_G)\cdot \log\frac{1}{p}\}$

ESTIMATE $N(n, m, H)$!

Lower bound:

- Blow up each $v \in H$ to V_v, $|V_v| = n_v = \frac{n^{x_v}}{v_H}$
- Join V_u, V_w completely iff $uw \in H$



$v_F = \sum_{v \in H} n_v \leq n$ if $0 \leq x_v \leq 1$

$e_F = \sum_{uw \in H} n_u \cdot n_w \leq \frac{1}{v_H^2} \sum_{uw} n^{x_u + x_w} \leq m$

if $x_u + x_w \leq \log_n m := q$

$N(F, H) \geq \prod_{v \in H} n_v = \Theta\left(n^{\sum x_v}\right)$

$\Rightarrow N(n, m, H) = \Omega(n^{\alpha_q(H)})$,

where $\alpha_q(H)$ is the optimal sol. to

$$\boxed{\begin{array}{ll} \text{(LP)} \quad \text{MAX} \ \sum_{v \in H} x_v, & \text{given} \\ 0 \le x_v \le 1 & \forall v \\ x_u + x_w \le q & \forall uw \in H \end{array}}$$

<u>THM</u> (JOR, 04) $N(n, m, H) = \tilde{\Theta}(n^{\alpha_q(H)})$

$\Downarrow$ [Alon 81: $N(\infty, m, H) = \Theta(m^{\alpha_1(H)})$]

$$\alpha_q(H) = \begin{cases} \alpha_1(H) \cdot q, & 0 \le q \le 1 \\ 2\alpha_1 - v + (v - \alpha_1) q, & 1 \le q \le 2 \end{cases}$$

$\Downarrow$

$$N(n, m, H) = \begin{cases} \Theta(m^{\alpha_1(H)}), & m \le n/2 \\ \Theta(n^{2\alpha_1 - v} m^{v - \alpha_1}), & n \le m \le n^2 \end{cases}$$

$$\Longrightarrow \quad \mu_G = \begin{cases} \Theta\left(\min_H \Psi_H^{1/\alpha_1(H)}\right), & n^{-1/m_G} < p \leq n^{-1/\Delta_G} \\ \Theta(n^2 p^{\Delta_G}), & p \geq n^{-1/\Delta_G} \end{cases}$$

<u>Ex.</u> $\alpha_q(K_3) = \frac{3}{2} q, \quad 0 \leq q \leq 2$

$$N(n, m, K_3) = \Theta(m^{3/2})$$

$$\mu_{K_3} = \Theta(n^2 p^2)$$

$$P(X_{K_3} \geq (1+\varepsilon)\mu) \doteq \exp\{-\Theta_\varepsilon(n^2 p^2)\}$$

$$P(X_{K_3} \leq (1-\varepsilon)\mu) = \begin{cases} \exp\{-\Theta_\varepsilon(n^3 p^3)\}, & p \leq n^{-\frac{1}{2}} \\ \exp\{-\Theta_\varepsilon(n^2 p)\}, & p \geq n^{-\frac{1}{2}} \end{cases}$$

All of above carries over to HYPERGRAPHS

$H=(V,E)$, $E \subseteq 2^V$

Friedgut, Kahn 98 : $N(\infty, m, H) = \Theta(m^{\alpha_1(H)})$

Dudek, Polcyn, R. :

$G^{(k)}(n,p)$ – a k-uniform random hypergraph

(H is k-uniform if $E \subseteq \binom{V}{k}$)

$$M_G = \max\left\{ m \le \binom{n}{k} : \forall H \subseteq G \ N(n,m,H) \le \mu_H \right\}$$

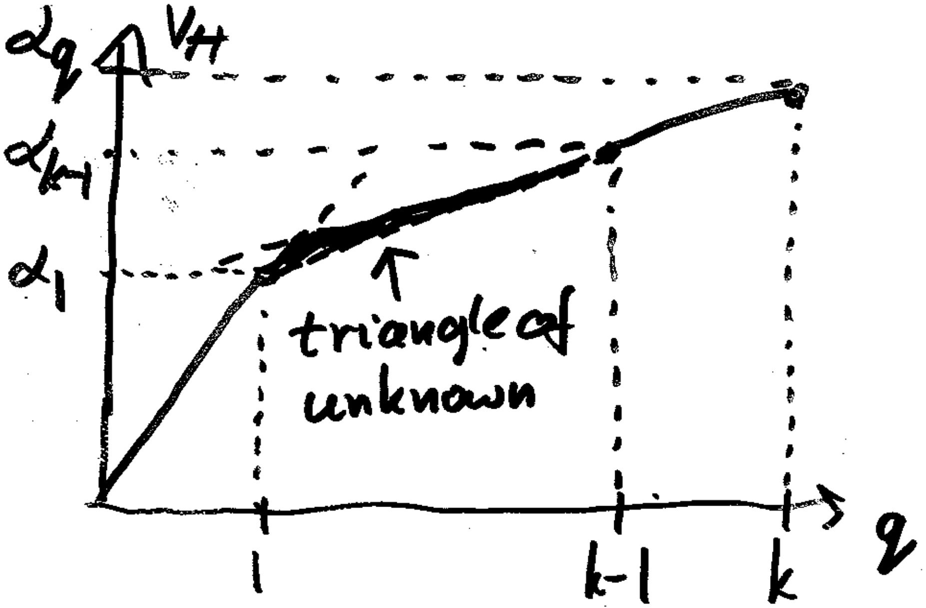
$$P(X_G \ge (1+\varepsilon)\mu_G) \doteq \exp\{-\Theta_\varepsilon(M_G)\}$$

$$N(n,m,H) = \Theta(n^{\alpha_q(H)}), \quad q = \log_n m$$

$$\boxed{\alpha_q(H) = MAX \sum x_v : 0 \le x_v \le 1, \ \sum_{v \in e} x_v \le q \quad \forall e \in H}$$

A KNOWLEDGE GAP:

$$\alpha_q(H) = \begin{cases} \alpha_1(H)\cdot q & , 0 \le q \le 1 \\ ? & \\ k\alpha_{k-1} - (k-1)v + (v-\alpha_{k-1})q & , k-1 \le q \le k \end{cases} \qquad (*)$$



Chvatal
$\Downarrow$
$\alpha_q(H)$ is piecewise linear, concave

$$\frac{v}{k}q \le \alpha_q \le v - \frac{(k-q)e}{\Delta}$$

$$\Downarrow$$

If H is regular (or $H \overset{\text{span}}{\supseteq} H'$: each component of H' is regular) then

$$\alpha_q(H) = \frac{v}{k}\cdot q$$

$\ell(H) = \#$ line segments of $d_q(H)$

$\tau(k) = \max\{\ell(H) : k\text{-uniform } H\}$

$\tau(2) = 2$, $\tau(3) \geqslant 6$

$\tau(k) \geqslant 2k-2 \quad \forall\, k \geq 4$

<u>CONJECTURE</u> $\tau(k) = \infty \quad \forall\, k \geqslant 3$