Lattice Point Enumeration, Linear Extensions, and the Theory of Partitions

Happy Birthday, Tom!

Carla D. Savage
Computer Science
North Carolina State University

Linear Diophantine Enumeration

Given constraints:

$$\mathcal{C} : c_{i,0} + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots + c_{i,n}\lambda_n \geq 0; \quad 1 \leq i \leq r$$

$S_{\mathcal{C}}$: nonnegative integer solutions : $\quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$

$$F_{S_{\mathcal{C}}}(x_1, x_2, \ldots, x_n) = \sum_{\lambda \in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

Generating function of $S_{\mathcal{C}}$:

$$F_{S_{\mathcal{C}}}(q) = F_{S_{\mathcal{C}}}(q, q, \ldots, q) = \sum_{\lambda \in S_{\mathcal{C}}} q^{|\lambda|}$$

where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is the weight of λ.

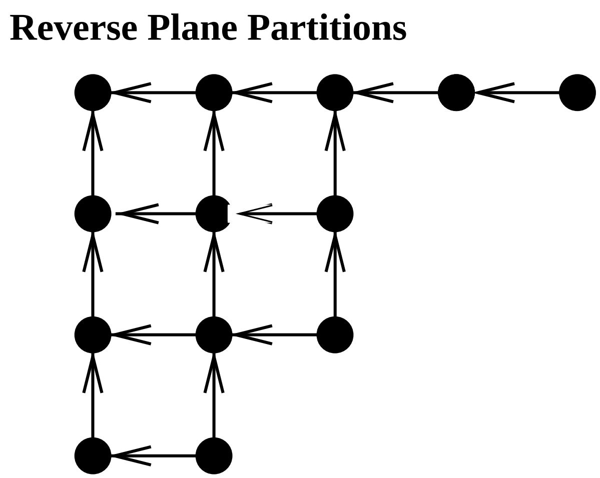
Coeff. of q^M is number of solutions of weight M

EXAMPLES

Let S be the set of nonnegative integer sequences $(\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$\begin{aligned} \lambda_1 - \lambda_2 - \lambda_3 &> 0 \\ \lambda_2 - \lambda_3 &\geq 0 \\ -\lambda_1 + \lambda_2 + 2\lambda_3 &\geq 0 \end{aligned}$$

$$F_S(x_1, x_2, x_3) = \frac{x_1^3 x_2 x_3}{(1 - x_1^3 x_2 x_3)(1 - x_1 x_2)(1 - x_1^2 x_2 x_3)}$$



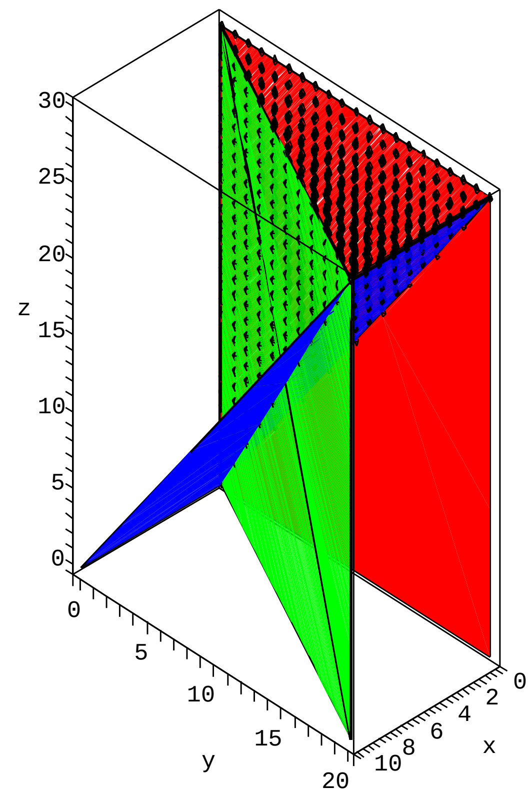
Let S be the set of vertex labelings λ by nonnegative integers satisfying: $u \to v \implies \lambda_u \geq \lambda_v$

$F_S(q)$:

$$\frac{1}{(1-q)^3(1-q^2)^3(1-q^3)(1-q^4)^2(1-q^5)^2(1-q^7)(1-q^8)}$$

[MacMahon 1912]

$\mathcal{C}$: $\lambda_1 \geq \frac{3}{2}\lambda_2$; $\lambda_2 \geq 2\lambda_3$; $\lambda_2 \geq 0$



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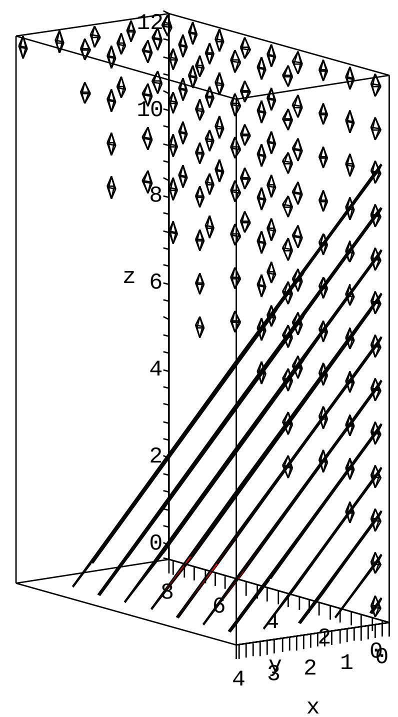


$$F_S(q) = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 4q^7 + 5q^8 + 6q^9 + \cdots$$

THEORIES

For simplicial cone defined by:

$$\mathcal{C} : c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots + c_{i,n}\lambda_n \geq 0; \quad 1 \leq i \leq n$$

$\exists$ an $n \times n$ nonnegative integer matrix $B = [b_{ij}]$ s. t.

$$F_{S_{\mathcal{C}}}(x_1, \ldots, x_n) = \frac{p(x_1, \ldots, x_n)}{\prod_{i=1}^{n}(1 - x_1^{b_{1i}} x_2^{b_{2i}} \cdots x_n^{b_{ni}})}$$

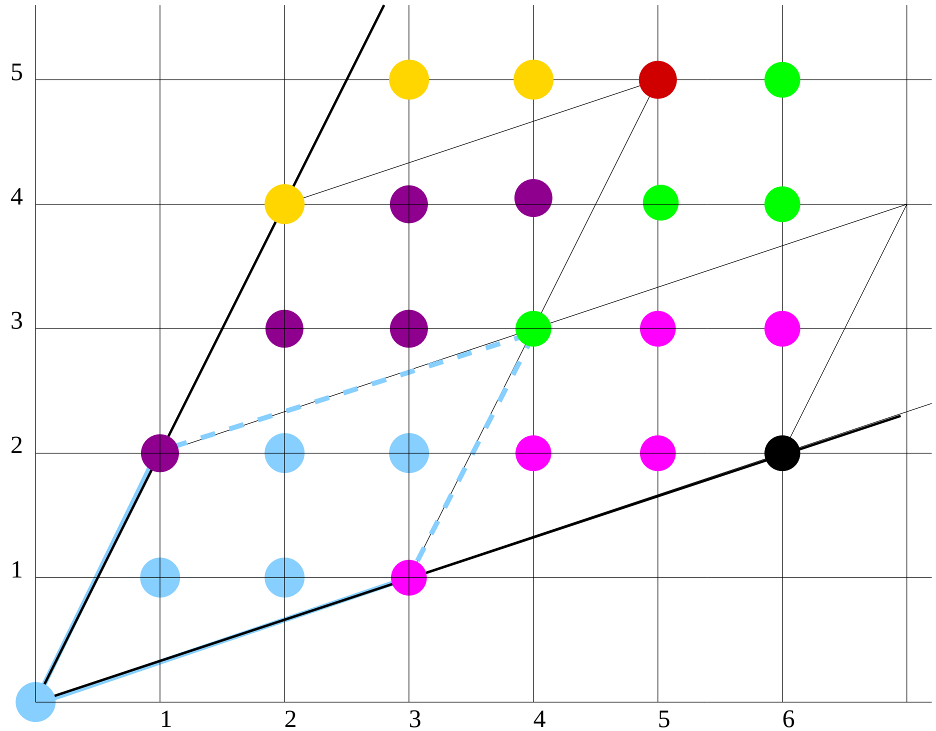
where $p(x_1, \ldots, x_n)$ is a polynomial and

$$p(1, 1, \ldots, 1) = det(B)$$

For fixed n, $det(B)$ can be exponential in the "size"

However, for fixed n, $F_{S_{\mathcal{C}}}$ admits a "short" representation, i.e. polynomial in problem size [Barvinok 1993].

Illustration: $2\lambda_1 - \lambda_2 \geq 0;$ $-\lambda - 1 + 3\lambda_2 \geq 0$



generators:

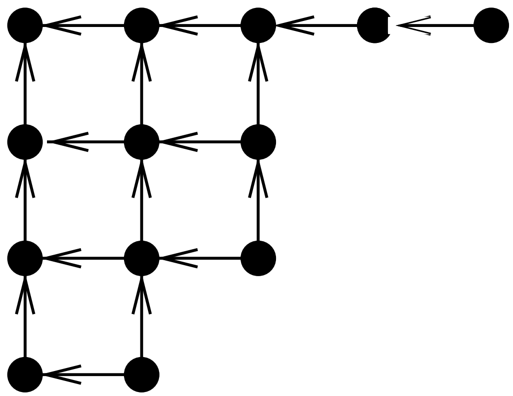
$[3,1]$ $[1,2]$

$$det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = 5$$

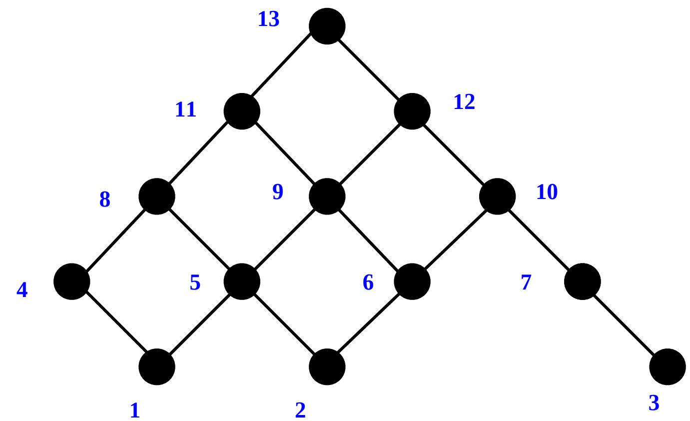
$$F_S(x_1, x_2) = \frac{1 + x_1x_2 + x_1^2x_2 + x_1^2x_2^2 + x_3^3x_2^2}{(1 - x_1^3x_2)(1 - x_1x_2^2)}$$

(1) Enumerating sequences S_D defined by digraph D to

(2) Counting linear extentions of poset P defined by D



D



P

linear extension π:

3 7 2 6 10 1 5 4 9 8 12 11

descents: **2 + 5 + 7 + 9 + 11**

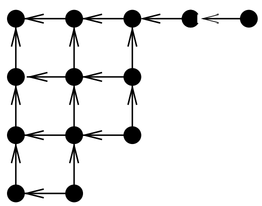
$$maj(\pi) = 2 + 5 + 7 + 9 + 11 = 34$$

Stanley's P-partitions relates

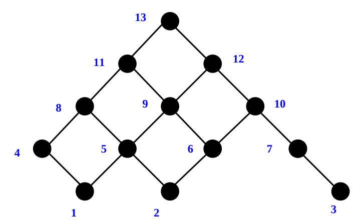
(1) Enumerating sequences S_D defined by digraph D to

(2) Counting linear extentions of poset P defined by D

D

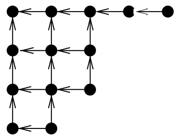


P

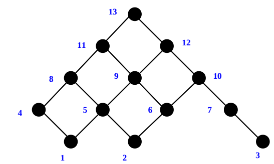


$$F_{S_D}(q) = \frac{\sum_{\pi \in L(P)} q^{maj(\pi)}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

(But (2) is #P-complete [Brightwell, Winkler 1991])

D 

So, to compute $|L(P)|$:

P 

$$F_{S_D}(q)(1-q)(1-q^2)\cdots(1-q^{13}) \quad = \sum_{\pi \in L(P)} q^{maj(\pi)}$$

$$= \frac{(1-q)(1-q^2)\cdots(1-q^{13})}{(1-q)^3(1-q^2)^3(1-q^3)(1-q^4)^2(1-q^5)^2(1-q^7)(1-q^8)}$$

Take limit as $q \longrightarrow 1$:

$$|L(P)| \quad = \frac{13!}{1*1*1*2*2*2*3*4*4*5*5*7*8}$$

(the hook length formula [Frame, Robinson, Thrall 1954])

Partition Theory

Given a set B of positive integers, define $p_B(n)$ by

$$\sum_{n=0}^{\infty} p_B(n)q^n = \prod_{b\in B} \frac{1}{1-q^b}$$

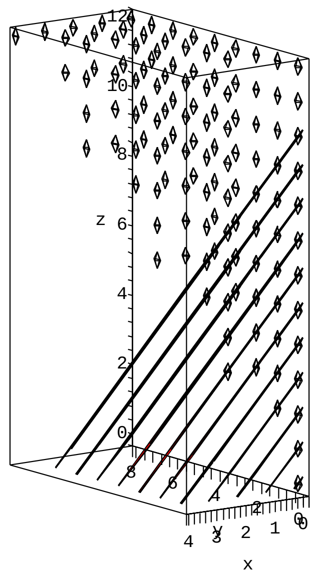
Then $p_B(n)$ is the number of partitions of the integer n into parts in B.

Example

$$\frac{1}{(1-q)(1-q^3)(1-q^5)} = (1+q+q^2+\cdots)(1+q^3+q^6+\cdots)(1+q^5+q^{10}+\cdots)$$

$$= 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 4q^7 + 5q^8 + 6q^9 + \cdots$$

Recall and compare: $\lambda_1 \geq \frac{3}{2}\lambda_2;$ $\lambda_2 \geq 2\lambda_3;$ $\lambda_2 \geq 0$



$$
\begin{aligned}
F_S(q) &= 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 4q^7 + 5q^8 + 6q^9 + \\
&= \frac{1}{(1-q)(1-q^3)(1-q^5)}
\end{aligned}
$$

SOFTWARE AND EXPERIMENTS

Software

LattE [De Loera, Haws, Hemmecke, Huggins, Tauzer, Yoshida]

- Code for Lattice point Enumeration, among other things; Runs on linux, uses Maple; can compute $F_{\mathcal{C}}(q)$, given linear constraints $\mathcal{C}$; implements Barvinok; polynomial for fixed dim.

posets [Stembridge]

- Package of 40 Maple programs; can compute $F_{S_D}(q)$ for a digraph D, count/generate linear extensions of a poset P, many other things; polynomial in poset width

Omega Package [Andrews, Paule, Riese]

- Mathematica program, based on MacMahon's Partition Analysis; given linear constraints C, computes $F_{\mathcal{C}}(x_1, \ldots, x_n)$; exponential, but potential to exploit recursive structure

Integer-sided triangles: nonneg. $(\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$\lambda_i + \lambda_j \geq \lambda_k \quad \text{for all perms } ijk \text{ of } 123$$

Generalizations, experiments with Omega [Andrews, Paule, and Riese 2001]

We propose: given integer sequence $(a_1, \ldots, a_n)$,

$$\mathcal{C}: \quad \sum_{i=1}^{n} a_i \lambda_{\pi(i)} \geq 0, \quad \text{all } \pi \in S_n$$

(wlog $a_1 \leq a_2 \leq \ldots \leq a_n$)

Hard for Omega as a_i grow.

Hard for Omega and LattE as n grows ($n!$ constraints).

For symmetrically constrained compositions

$$\mathcal{C}: \quad \sum_{i=1}^{n} a_i \lambda_{\pi(i)} \geq 0, \quad \text{all } \pi \in S_n$$

Theorem [Gessel, Lee, S 2008] if $\sum_{i=1}^{n} a_i = 1$,

$$F_{\mathcal{C}}(q) = \frac{\sum_{\pi \in S_n} \prod_{j \in D_\pi} q^{j-n\sum_{i=1}^{j} a_i}}{(1-q^n)\prod_{j=1}^{n-1}(1-q^{j-n\sum_{i=1}^{j} a_i})}.$$

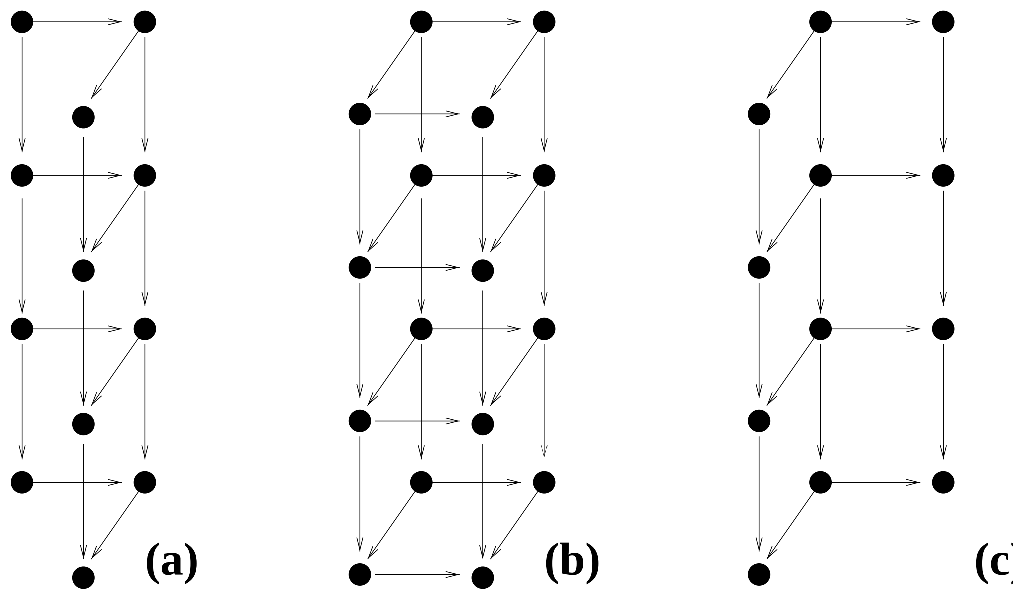
Can compute via dynamic programming.

For some a_i, can simplify using:

$$\sum_{\pi \in S_n} q^{maj(\pi)} x^{des(\pi)} = \prod_{i=0}^{n}(1-xq^i)\sum_{k=0}^{\infty}([k+1]_q)^n x^k.$$

[Carlitz 1954]

[S. Singanallur] Linear extensions



(a) $2(3n)!/n!/(n+1)!/(n+2)!$ [$3 \times n$ plane partitions]

(b) $2 \times 2 \times n$ solid partitions

(c) $4^n(3n)!/(n+1)!/(2n+1)!$‘ [Kreweras 1965]

All hard for LattE and Omega as n grows.

$2 \times 2 \times n$ solid partitions [S. Singanallur]: **recurrence for the generating function using partition analysis.**

Still, "posets" package is much faster:

```
>> 1    2x2x1   2
>> 2    2x2x2   48
>> 3    2x2x3   2452
>> 4    2x2x4   183958
>> 5    2x2x5   17454844
>> 6    2x2x6   1941406508
>> 7    2x2x7   242201554680
>> 8    2x2x8   32959299267334
>> 9    2x2x9   4801233680739724
>> 10   2x2x10  738810565910888784
>> 11   2x2x11  118929992674840615128
>> 12   2x2x12  19880920716640427983476
>> 13   2x2x13  3431624482227380273056728
>> 14   2x2x14  608880419873586515669564728
>> 15   2x2x15  110654016191338341346670548240
>> 16   2x2x16  20536574090713344110860752530646
>> 17   2x2x17  3882925024331174796857101684510428
>> 18   2x2x18  746410931448945012196513727291312844
>> 19   2x2x19  145626362670805760264809414243057616552
>> 20   2x2x20  28794547473359904233269297596817899967540
```

Sequences constrained by the ratio of consecutive parts

$$\frac{\lambda_1}{a_1} \geq \frac{\lambda_2}{a_2} \geq \cdots \geq \frac{\lambda_{n-1}}{a_{n-1}} \geq \frac{\lambda_n}{a_n} \geq 0$$

Recurrence [Corteel, S 2005]

$\longrightarrow \quad O(n \sum a_i)$ algorithm [Wong 2008]

Experiments [Iyer, Wong]

LattE shows exponential growth with n, even for, e.g.

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \frac{\lambda_3}{1} \geq \frac{\lambda_4}{2} \geq \cdots \geq 0$$

(Wong's algorithm is $O(n)$ in this case.)

RESULTS

Lecture Hall Partitions

$$L_n : \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_n}{1} \geq 0$$

LPE says:

$$F_{L_n}(q) \;=\; \frac{f(q)}{(1-q^n)(1-q^{n+(n-1)})\cdots(1-q^{n+\cdots+1})},$$

where $f(1) = (n-1)!$

Lecture Hall Theorem [Bousquet-Mélou, Eriksson 1997]

$$F_{L_n}(q) = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}$$

Slicing, truncating, "binomial essence" emerges:

$$L_{n,k}^{(j)}: \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} \geq 0$$

Theorem [Corteel, S 2005, 2006]:

$$|L_{n,k}^{(j)}| = j^k \binom{n}{k}$$

$$L_{n,k}(q) = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2n-k+1};q)_k}$$

where $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$

Theorem [Corteel, S 2008]

$$\sum_{j=0}^{\infty} q^{j(3j-1)/2} \frac{(q^2;q^6)_j}{(q;q)_{3j}} = \frac{1}{(q;q^3)_\infty (q^5;q^6)_\infty}$$

and

$$\sum_{j=0}^{\infty} q^{j(3j+1)/2} \frac{(q^4;q^6)_j}{(q;q)_{3j+1}} = \frac{1}{(q^2;q^3)_\infty (q;q^6)_\infty}$$

Proof. Show both sides of first count λ satisfying

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots$$

and both sides the second count λ satisfying

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{2} > \frac{\lambda_3}{1} > \frac{\lambda_4}{2} > \cdots.$$

Refinement

For λ satisfying

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots$$

the generating function

$$F(x, y, x, y, \ldots) = \sum_{\lambda} x^{(\lambda_1+\lambda_3+\ldots)} y^{(\lambda_2+\lambda_4+\cdots)}$$

computed in two different ways gives

$$\sum_{j=0}^{\infty} \frac{x^{j^2} y^{j(j-1)/2} (-x; x^2 y)_j}{(x^2; x^2 y)_j (x^2 y; x^2 y)_j} = \frac{1}{(x; x^2 y)_\infty (x^4 y; x^4 y^2)_\infty}$$

Euler's Partition Theorem:

The number of partitions of an integer N into odd parts is equal to the number of partitions of N into distinct parts.

Example: $N = 8$

Odd parts:

(7,1) (5,3) (5,1,1,1) (3,3,1,1) (3,1,1,1,1,1) (1,1,1,1,1,1,1,1)

Distinct Parts:

(8) (7,1) (6,2) (5,3) (5,2,1) (4,3,1)

Euler's Partition Theorem:

The number of partitions of an integer N into odd parts is equal to the number of partitions of N into distinct parts.

This is $\lim_{n\to\infty}$ (The Lecture Hall Theorem) since

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_n}{1} \geq 0 \rightarrow \text{partitions into distinct parts}$$

and

$$\prod_{i=1}^{n} \frac{1}{1-q^{2i-1}} \rightarrow \text{partitions into odd parts}$$

ℓ-sequences

For integer $\ell \geq 2$, define the sequence $\{a_n^{(\ell)}\}_{n\geq 1}$ by

$$a_n^{(\ell)} = \ell a_{n-1}^{(\ell)} - a_{n-2}^{(\ell)},$$

with initial conditions $a_1^{(\ell)} = 1$, $a_2^{(\ell)} = \ell$.

$$\{a_n^{(3)}\} \;=\; 1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377,\ldots$$

$$\{a_n^{(2)}\} \;=\; 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ldots$$

The ℓ-Lecture Hall Theorem [BME 1997]: For ℓ-seq. a_i,

$$L_n: \quad \frac{\lambda_1}{a_n} \geq \frac{\lambda_2}{a_{n-1}} \geq \ldots \geq \frac{\lambda_{n-1}}{a_2} \geq \frac{\lambda_n}{a_1} \geq 0$$

$$F_{L_n}(q) = \frac{1}{(1-q^{a_0+a_1})(1-q^{a_1+a_2})(1-q^{a_2+a_3})\cdots(1-q^{a_{n-1}+a_n})}$$

Taking limits gives:

The ℓ-Euler Theorem [BME 1997]: The number of partitions of an integer N into parts from the set

$$\{a_0+a_1,\ a_1+a_2,\ a_2+a_3,\ \ldots\}$$

is the same as the number of partitions of N in which the ratio of consec. parts is $> c_\ell = \frac{\ell+\sqrt{\ell^2-4}}{2}$.

Bijection for the ℓ-Euler Theorem [Yee,S 2008]

Given a partition μ into parts in

$$\{a_0 + a_1,\ a_1 + a_2,\ a_2 + a_3,\ \ldots\}$$

construct $\lambda = (\lambda_1, \lambda_2, \ldots)$ by inserting the parts of μ in non-increasing order as follows:

To insert $a_{k-1} + a_k$ into $(\lambda_1, \lambda_2, \ldots)$:

If $k = 1$, then add a_1 to λ_1;

otherwise, if $(\lambda_1 + a_k - a_{k-1}) > c_\ell(\lambda_2 + a_{k-1} - a_{k-2})$,

add $a_k - a_{k-1}$ to λ_1, add $a_{k-1} - a_{k-2}$ to λ_2;

recursively insert $a_{k-2} + a_{k-1}$ into $(\lambda_3, \lambda_4, \ldots)$

otherwise,

add a_k to λ_1, and add a_{k-1} to λ_2.