Some recent progresses on Random Bernoulli Matrices

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M_n $n \times n$ matrix, with entries ξ_{ij} being iid random variables:

$$\mathbf{P}(\xi_{ij} = 1) = \mathbf{P}(\xi_{ij} = -1) = 1/2.$$

Questions. Singularity probability, Determinant, Permanent, Special singular values, etc.

Methods extend to may other distributions.

In all questions/results: $n \to \infty$.

Question 1. What is p_n, the probability that M_n is singular ?

Conjecture. $p_n = (1/2 + o(1))^n$.

Lower bound is trivial: probability of two rows of columns are equal (up to sign).

Upper bound: Komlós (67) $o(1)$, Komlós (late 70s) $O(n^{-1/2})$, Kahn-Komlós-Szemerédi (95): $.999^n$, Tao-V. (04): $.952^n$, Tao-V. (05): $(3/4 + o(1))^n$, Bourgain-V.-Wood (07): $(1/\sqrt{2} + o(1))^n$.

Tools. Switching technique (KKS), Inverse Theorems (TV05), Fractional dimension (BVW 07).

Question 2. What is the typical value of Det M_n ?

$\mathbf{E}(\text{Det } M_n) = 0$ (symmetry); $|\text{Det } M_n| \le n^{n/2}$ (Hadamard's bound).

$\mathbf{E}(\text{Det }^2 M_n) = n!$ (Turán)

Det $M_n = \sum_{\sigma \in S_n} (-1)^{sign(\sigma)} \prod_i \xi_{i\sigma_i}$

Det $^2 M_n = \sum_{\sigma,\sigma'} \prod_i \xi_{i\sigma_i} \xi_{i\sigma'(i)}$

$\mathbf{E}(\text{Det }^2 M_n) = \sum_{\sigma,\sigma'} \mathbf{E} \prod_i \xi_{i\sigma_i} \xi_{i\sigma'(i)} = n!$

Conjecture. With probability tending to 1,

$$|\text{Det } M_n| \approx \sqrt{n!} = n^{(1/2-o(1))n}.$$

Theorem. (Tao-V. 04) With probability tending to 1,

$$|\text{Det } M_n| = n^{(1/2 - o(1))n}.$$

Proof 1. Let $X_1 \ldots, X_n$ be the rows of M_n. $|Det M_n|$ is the volume of the parallelepiped spanned by these vectors. So

$$|\text{Det } M_n| = \prod_i distance(X_i, Span(X_1, \ldots, X_{i-1})).$$

Main idea: For most i, $distance(i) \approx \sqrt{n-i}$. (Easy to check if the entries are normal instead of Bernoulli.)

Proof 2. $|\text{Det } M_n| = \prod_{i=1}^n s_i$, where s_i are the singular values. The limiting distribution of s_i is known. Moreover (and critically), the last few singular values are NOT too small (at least n^{-C}, Tao-V. 05).

Permanent. Per $M_n = \sum_{\sigma \in S_n} \prod_i \xi_{i\sigma_i}$.

Similar to Det M_n, $\mathbf{E}$(Per M_n) $= 0$ (symmetry) and $\mathbf{E}$(Per $^2 M_n$) $= n!$

Conjecture. With probability tending to 1,

$$|\text{Per } M_n| \approx \sqrt{n!} = n^{(1/2 - o(1))n}.$$

It was not known, however, that Per M_n is non-zero with probability tending to one.

Alon (also Wanless) observed that if $n + 1 = 2^m$, then Per of any ± 1 matrix equals 2^{m-1} mod 2^m.

Theorem. (Tao-V. 08) With probability tending to 1,

$$|\text{Per } M_n| = n^{(1/2-o(1))n}.$$

In particular Per $\neq 0$ with probability $1 - n^{-c}$, for some constant c.

Main difficulty. Cannot find good geometric or linear algebraic interpretation as with Determinant.

This leaves combinatorics the (seemingly) only option. Upper bound is trivial by second moment method. We need to show lower bound only.

Our plan. Expose the matrix row by row. At every step try to keep a large family of full size minors with large permanents.

When we say a k-minor, we understand that it is formed by the first k rows of M_n. Thus, a k-minor is identified by a subset of size k of $[n] = \{1, \dots, n\}$.

Fact 1. If there are l $(n-1)$-minors each has permanent at least λ (in absolute value), then $|\text{Per } M_n| \geq \lambda$ with probability $1 - O(1/\sqrt{l})$.

If A is a $k \times k$ matrix with entries ξ_i and the first rows, and A_i is the $(k-1) \times (k-1)$ submatrix obtained by deleting the first row and the ith column, then

$$\text{Per } A = \sum_{i=1}^{k} \xi_i \text{Per } A_i.$$

Littlewood-Offord-Erdős lemma. Let $v_1, \ldots, v_m$ are real numbers, at least l have absolute values ≥ 1. Let $\xi_1, \ldots, \xi_m$ be iid Bernoulli random variables and $S := x_1 v_1 + \ldots + \xi_m v_n$. Then

$$\mathbf{P}(|S| \leq 1) = O(1/\sqrt{l}).$$

More generally $\mathbf{P}(|S| \leq x) = O(x/\sqrt{l})$.

Definition. $E_{k,N,\lambda}$ denotes the event that there are N $k \times k$ minors with permanents at least λ in absolute value. (This event is defined by the first k rows.)

Goal 1. With high probability, $E_{n,1,n^{(1/2-\epsilon)n}}$ holds.

By Fact 1, it is sufficient to have

Goal 2. With high probability, $E_{n-1,m,n^{(1/2-\epsilon)n}}$ holds for some $m \to \infty$.

What we do have, for free, is

Starting point. With probability 1, $E_{1,n,1}$ holds.

Main ideas.

In the bulk of the process, there are three types of steps:

λ-growing. In most of the steps, the permanent (λ) increases by almost $\sqrt{n}$, while the size of the family drops by at most a constant factor.

N-growing. In a constant fraction of the remaining steps, the permanent stays at least the same, but the size (N) of the family increases by a huge factor ($n^{\Omega(1)}$).

Maintaining. In the rest of the steps, the permanent stays at least the same, and the size of the family drops by at most a constant factor.

At the end of the process, we allow both λ and N to decrease, but at only very few steps (*End Game*).

Lemma 1. (Maintaining) Assume $1 \leq k \leq (1-\epsilon)n$, then

$$\mathbf{P}(E_{(k+1),1,\lambda} | E_{k,1,\lambda}) = 1 - \exp(-\Omega(\epsilon n)).$$

More generally,

$$\mathbf{P}(E_{(k+1),N/2,\lambda} | E_{k,N,\lambda}) = 1 - \exp(-\Omega(\epsilon n)).$$

Proof. Each (rich) parent has $n - k \geq \epsilon n$ children, each of whom can be rich with probability at least 1/2 (independently)

$$\text{Per } A = \sum_{i=1}^{k} \xi_i \text{Per } A_i.$$

Lemma 2. (Growing) Assume $\epsilon n \leq k \leq (1-\epsilon)n$. One can partition $E_{k,N,\lambda} = E'_{k,N,\lambda} \cup E^{''}_{k,N,\lambda}$ such that for some sufficiently small constant $\delta > 0$

$$\mathbf{P}(E_{k+1,\epsilon N/6,\lambda n^{1/2-\delta}}|E') \geq 1 - o(1)$$

$$\mathbf{P}(E_{k+1,Nn^{\delta},\lambda}|E^{''}) \geq 1/3.$$

Observation. Let L be the number of N-growing steps:

$$n^{\delta L}(\epsilon/6)^n \leq 2^n$$

so $L = o(n)$. Thus, in most steps, we do increase λ by $n^{1/2-\delta}$ (with high probability). (One can use a sub-martingale argument to make this rigorous.)

Lemma 3. (End-game) Set $k_1 = (1-\epsilon)n$; $L = \log n/100$.

$$\mathbf{P}(E_{n-L,n^{.99},\lambda}|E_{k_1,1,\lambda}) = 1 - o(1)$$

Furthermore, the sets in the family has disjoint complements. Finally, for all $1 < j \le L$ (it is important to have strictly > 1 here):

$$\mathbf{P}(F_{n-j+1,\frac{N}{10},\frac{\lambda}{n}}|F_{n-j,N,\lambda}) = 1 - n^{-\Omega(1)}.$$

Here F is E with the additional assumption of disjoint complements.

If all lemmas hold, then we have, with high probability

$$F_{n-1,\frac{n^{.99}}{10^L},\frac{n^{(1/2-\epsilon)n}}{n^L}} = F_{n-1,n^{.9},n^{(1/2-\epsilon-o(1))n}}$$

which guarantees **Goal 2**.

Lemma. (First moment)Let I_j, $1 \le j \le m$ be indicator variables with expectation at least $1-\delta$ each, then $\mathbf{P}(\text{more than } cm \text{ fail}) \le \delta/c$.

Proof of Growing Lemma.

Look at the bipartite graph G between parents and children:

$$\epsilon nN \le e(G) = N(n-k) = \sum_{l=1}^{n} l|V_l|$$

where V_l is the set of children with exactly l parents.

Set $K = n^{1-\delta}$, for some small δ.

$$\epsilon nN \le K(|V_1| + \ldots + |V_K|) + n(V_{K+1} + \ldots + |V_n|).$$

So there are two cases:

(1) $|V_{K+1}| + \ldots + |V_n| \ge \epsilon N/2$

(2) Not (1), so: $|V_1| + \ldots + |V_K| \ge \epsilon N n^{\delta}/2.$

If (1), then each child in $V_{K+1} \cup \ldots \cup V_n$ has at least K λ-rich parents. Recall that $K = n^{1-\delta}$. By L-O-E lemma, each child will be $n^{1/2-\delta}\lambda$-rich with probability $1 - O(n^{-\delta/2})$. Now use the First Moment Lemma.

If (2), then each child in $V_1 \cup \ldots \cup V_K$ has at last one λ-rich parent, so he/she is λ-rich with probability 1/2. But now the number of children is huge: $\Omega(Nn^{\delta})$. Use the First Moment Lemma.

Proof of End-Game Lemma. We show the second (more critical) part.

By the condition $F_{n-j,N,\lambda}$, there are sets $A_1, \ldots, A_N$ of size $n-j$ with disjoint complements, corresponding to the large minors. Set $B_i = A_i \cup \{h_i\}$, for an arbitrary $h_i \in \bar{A}_i$. Look at the $n-j+1$ -minors defined by the B_i.

Set $T = n^{\cdot 1}$. Call a child B_i *good* if it has at least T λ/n-rich parents. Notice that one such parent (namely A_i) always exists.

(1) There are $N/2$ good children. By L-O-E lemma, each good child has probability $1 - O(1/\sqrt{T})$ to be λ/n-rich. Use First Moment Lemma.

(2) There are $N/2$ bad children. Let I be the set of $\widehat{\text{b}}$ad i and H be the set of h_i, $i \in I$. Draw a bipartite graph by connecting i with $h_{i'}$ if $h_{i'} \in B_i$ and the minor formed by $B_i \backslash h_{i'}$ is λ/n -rich.

Since B_i is bad, the degree of i is at most T. By double counting, the number of i such that degree of h_i is at least $2T$ is at most $N/4$. Thus the set

$$I' := \{i | \deg\, h_i \leq 2T\}$$

has at last $N/4$ elements. We will condition on the entries outside I'.

For $i \in I'$, define

$$Y_i := \min\{\frac{|\text{Per } B_i|}{\lambda}, 1\}; \; Y = \sum_{i \in I'} Y_i.$$

Since $\mathbf{E}(Y_i) \geq 1/2$, $\mathbf{E}Y \geq |I'|/2 \geq N/8$.

We are going to finish by Azuma, showing that Y is strongly concentrated. The key is to bound the Lipschitz coefficient of Y.

Consider ξ_h, $l \in I'$. Flip ξ_h and see how much it influence Y_i:

(1) If $h \notin B_i$, then influence is zero.

(2) If $h \in B_i$ but its minor is not λ/n-rich, then the influence is at most $2/n$.

(3) If $h \in B_i$ and its minor is λ/n-rich, then the influence can be as large as 1.

On the other hand, the number of indices i where the (3) case can occur is at most $2T$ by the definition of I'. So the influence of ξ_h on Y is at most

$$n \times (2/n) + 2T \leq 3T << N.$$

Azuma's inequality shows that w.h.p, Y is close its mean, in particular, $Y \geq N/9$, which implies there are at least $N/10$ λ/n-rich B_i.

Open Question 1. $q_n := \mathbf{P}(\text{Per } M_n = 0.$? $q(n)$.

The truth may be even super exponential, $q(n) \leq n^{-cn}$ for some constant $c > 0$. But even exponential bound seems very hard.

Our method gives $q(n) \leq n^{-\Omega(1)}$.

Open Question 2. Limiting distribution ?

For determinant of random matrices with normal entries, it is not hard to show

$$\frac{\log \text{Det } M_n^2 - \log(n-1)!}{2\sqrt{\log n}} \to N(0,1).$$

Girko claimed the same for more general entries. Should one expect the same for permanent ?