Cycles

in

Sparse Graphs

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Outline

1 Extremal Graph Theory

- If $\mathcal{F}$ is a set of graphs, let ex$(n,\mathcal{F})$ denote the maximum size of an n-vertex graph containing no graph in $\mathcal{F}$ - the **Turán number** for $\mathcal{F}$.

- For example, if $\mathcal{F}$ consists of all cycles, i.e.

$$\mathcal{F} = \{C_3, C_4, C_5, \ldots\}$$

then ex$(n,\mathcal{F}) = n - 1$.

- Generally accepted that determining ex(n, C_{2k}) is very difficult, while

$$\text{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

- **Proposition**

  If $\mathcal{F}$ consists of all odd cycles,

  $$\mathcal{F} = \{C_3, C_5, C_7, \ldots\}$$

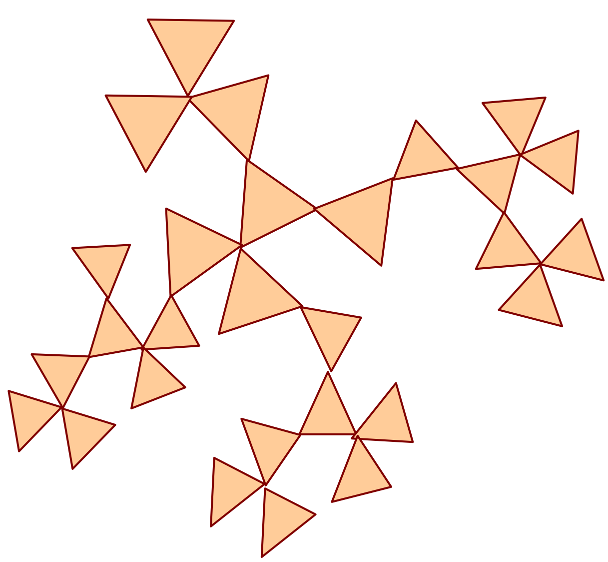
  then $\mathrm{ex}(n, \mathcal{F}) = \left\lfloor \frac{n^2}{4} \right\rfloor.$

- **Proposition**

  If $\mathcal{F}$ consists of all even cycles,

  $$\mathcal{F} = \{C_4, C_6, C_8, \ldots\}$$

  then $\mathrm{ex}(n, \mathcal{F}) = [\frac{3}{2}(n-1)].$



2 Cycles mod k

- **Theorem (Bollobás)**

  There exists $f(k)$ such that every graph of average degree at least $f(k)$ contains a cycle of length 0 mod k.

- The minimum $f(k)$ is not known.

- **Theorem (V)**

  Suppose l is odd or k is even. Then every graph of minimum degree at least 3k contains a cycle of length l mod k.

- **Conjecture (V)**

  $f(2k) = 2k - 1$ for $k > 1$

3 Unavoidable cycles

Conjecture (Erdös-Gyárfás)

Every graph of large enough average degree contains a cycle of length a power of two.

Theorem A (Sudakov-V)

Let Λ be an infinite increasing sequence of even integers that does not grow faster than the tower sequence

$$2 \qquad 2^2 \qquad 2^{2^2} \qquad 2^{2^{2^2}} \qquad \cdots$$

Then any n-vertex graph of average degree at least $\exp(8\log^* n)$ *contains a cycle of length in Λ.*

Construction

Starting with $G_0 = K_{4,4}$, let G_{r+1} be graph with

- average degree $d(r+1) = d(r)^{d(r)}$
- girth larger than $d(r+1)$
- order $n(r+1) = d(r+1)^{d(r+1)}$

With positive probability, random graphs with appropriate edge probability satisfy these criteria. Let

$$F = G_0 \sqcup G_1 \sqcup G_2 \sqcup \cdots \sqcup G_r$$
$$\Lambda = \{d(r) : r \geq 0\} = \{2^2, 2^8, 2^{2^{11}}, \ldots\}$$

and $n = |V(F)|$.
Observe that

- $e(F) \approx \dfrac{n \log n}{2 \log \log n}$
- $\mathcal{C}(F) \cap \Lambda = \varnothing$

and Λ grows faster than a tower, but not faster than

$$2^2 \qquad 2^{2^3} \qquad 2^{2^{2^4}} \qquad 2^{2^{2^{2^5}}} \quad \cdots$$

4 Cycle spectrum

- The **cycle spectrum** of a graph G, denoted $C(G)$, is the set of lengths of cycles in G.

$$C(G) = \{\ell : \exists C_\ell \subseteq G\}$$

- Let $\nu_k(g)$ be the **Moore Bound** for the number of vertices in a k-regular graph of girth g.

Conjecture (Erdös)

Let G be a graph of minimum degree at least k and girth at least g. Then

$$|\mathcal{C}(G)| = \Omega(\nu_k(g)) = \Omega(k^{[(g-1)/2]}).$$

Previous results
(Erdös, Faudree, Rousseau, Schelp; Zhang)

Girth	$\lvert\mathcal{C}(G)\rvert \geq \Omega(\gamma(k))$	$\nu_k(g)$
$g = 3$	$k - 2$	$k + 1$
$g = 4$	k	$2k$
$g = 5$	k^2	$k^2 + k + 1$
$g = 6$	k^2	$2(k^2 + k + 1)$
$g = 7$	$k^{5/2}$	$(k^3 + k^2 + k + 1)$
$g = 8$	$k^{5/2}$	$2(k^3 + k^2 + k + 1)$
g	$k^{g/8}$	$(k^4 + k^3 + k^2 + k + 1)$

Theorem B (Sudakov-V)

Let k be an integer and G a graph of average degree at least $48k$ and girth g. Then G contains cycles of $\Omega(\nu_k(g))$ consecutive even lengths.

Proof

Fact 1 (Local Expansion)

Let G be a graph of minimum degree at least $6d$ and girth more than g. Then

$$|\partial X| > 2|X| \qquad \forall X \subseteq V(G) : |X| > \frac{1}{3}[(d-1)^{\frac{g}{2}}].$$

Fact 2 (Pósa's Lemma)

If G is a graph and every $X \subseteq V(G)$ of size at most m has $|\partial X| > 2|X|$ then G contains a path of length 3m.

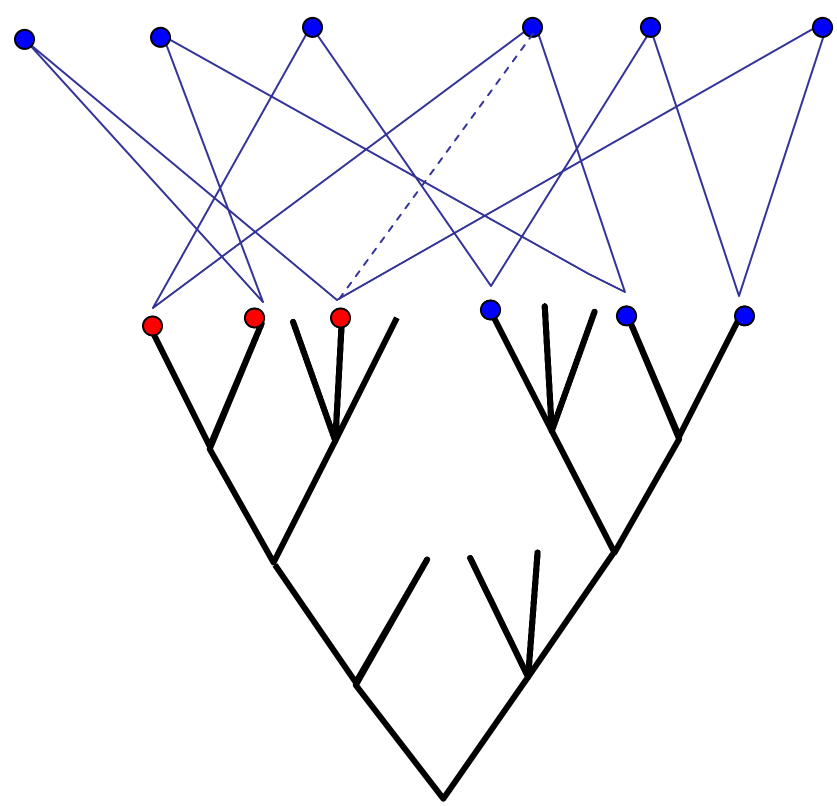
Let T be a breadth first search tree in a component of a graph G of average degree at least $48k$ and girth g.

By the pigeonhole principle, two consecutive levels of the tree induce a graph H of average degree at least $12k$.

Let F be a subgraph of H of minimum degree at least $6k$.

By Pósa's Lemma, F has a path of length at least $(k-1)^{g/2}$.

We can get a cycle of length at least $(k-1)^{g/2}$ with a chord.



5 Three conjectures

- **Conjecture 1.**

  Every graph of average degree at least $2k-1$ contains a cycle of length zero mod $2k$.

- **Conjecture 2 (Erdös)**

  Every triangle-free graph of chromatic number at least k contains a cycle of length $k^{2-o(1)}$.

- **Conjecture 3**

  Is there a sequence of odd numbers such that every graph of large enough chromatic number has a cycle of length in that sequence?

  Does every graph of chromatic number at least four have a cycle of length a power of two plus one?