

Bounds on the k -dimension of Products of Special Posets

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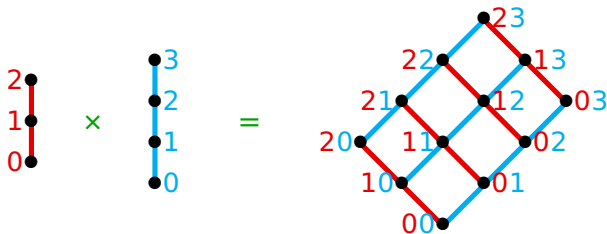
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Ex. $P = 3, Q = 4$.



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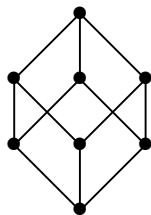
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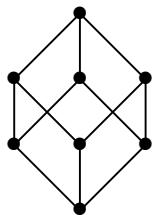
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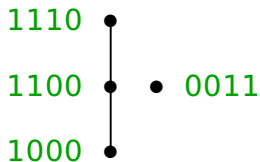
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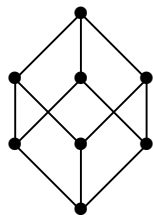
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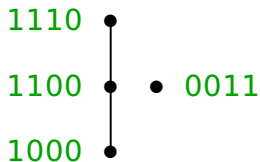
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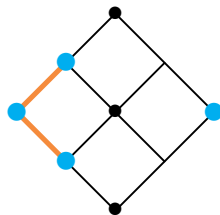
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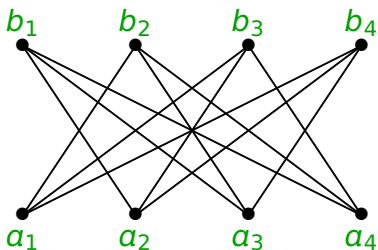
$\dim_2 P = 4$



$\dim_3 P = 2$

The Standard Example S_n

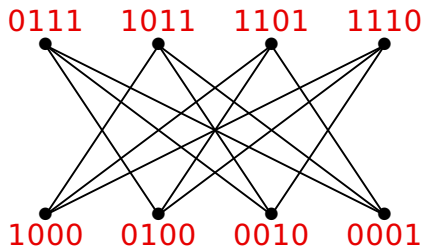
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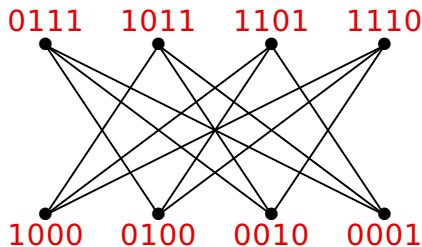
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Upper bound: $\dim_2 S_n \leq n$.



Lower bound: $\dim S_n \geq n$.

Each a_i must be above b_i on some extension.
Each extension has at most one such pair.

$$A - \{a_i\} < b_i < a_i < B - \{b_i\}$$

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Thm. (C. Lin [1991]) $\dim(S'_m \times S_n) = m + n - 2$,
where S'_m is S_m plus a global max and global min.

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Thm. $\dim_4(S_m \times S_n) = m + n - 2,$
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Thm. If P is a bounded poset and k is fixed, then
 $2 \dim_k mP - \dim_k(mP \times mP)$ is unbounded.

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Since $A_n \times A_n = A_{n^2}$,

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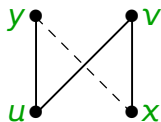
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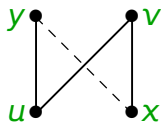
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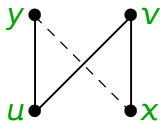


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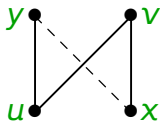
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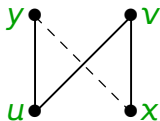
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- Suffices to handle (x, y) with x min'l and y max'l in R .

$S_m \times S_n$ - Critical Pairs and 2-extensions

Write $(a_{i,j}, b_{r,s})$ for $((a_i, a'_j), (b_r, b'_s))$ with $i, r \in [m]$ and $j, s \in [n]$. Note that $a_{i,j} < b_{r,s}$ if $i \neq r$ and $j \neq s$.

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Def. Critical pairs $(a_{i,j}, b_{r,s})$ are of three types:

vertical pairs - $i \neq r$ and $j = s$. $\exists m^2n - mn$ of these.

horizontal pairs - $i = r$ and $j \neq s$. $\exists mn^2 - mn$ of these.

straight pairs - $i = r$ and $j = s$. $\exists mn$ of these.

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Def. Types of 2-extensions ($m = 3$ and $n = 4$).

0	1	1	1
0	1	1	1
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1	1	1	1
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0	0	0	0

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1	0	1	1
0	1	1	1

1	0	0	0
1	0	0	0
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0	0	0	0
0	0	0	0
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0	0	0	0
0	0	0	0
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0	0	0	0
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V	$m^2 - m$	0	m	$m^2 - m$	$2n - 2$	m
H	0	$n^2 - n$	n	$2m - 2$	$n^2 - n$	n
B	$m - 1$	$n - 1$	1	$2m - 2$	$2n - 2$	2
C	2	2	0			
tot	$m^2n - mn$	$mn^2 - mn$	mn	$m^2n - mn$	$mn^2 - mn$	mn

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For $k \geq 4$, $V + H + B < m + n - 2 \Rightarrow V \leq n - 2$ or $H \leq m - 2$.
 If $V = n - r$, then vertical $\Rightarrow H + B \geq \frac{rm}{2} > m + r - 3$.

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$\dim_2(S_m \times S_n)$, again

Thm. For $m, n \geq 3$, $\dim_2(S_m \times S_n) = m + n$.

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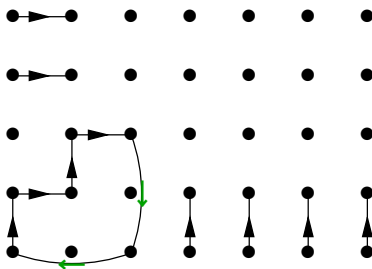
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Idea: For $\prod_{i=1}^t S_t$, present a cycle C_t of length t^2 in the t -dimensional grid consisting of 2-conflicted arrows. Yields $\dim_2(\prod_{i=1}^t S_t) = t \cdot t$.

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For $\prod_{i=1}^t S_{n_i}$ with each $n_i \geq t$, add $n_i - t$ arrows for extra hyperplanes in the i th direction, copying arrows on C_t . Yields $\dim_2(\prod_{i=1}^t S_{n_i}) = \sum_{i=1}^t n_i$.

The t -lagging cycle

Def. The t -lagging cycle is the cycle C_t consisting of vectors v_1, \dots, v_{t^2} in $[t]^t$ given by letting $v_1 = (1, \dots, 1)$ and, for $j \geq 1$, forming v_{j+1} from v_j by adding 1 (mod t) to each coordinate except the i th, where $i \equiv j \pmod t$.

Ex. C_4 , reading down columns:

(1, 1, 1, 1)	(4, 4, 4, 4)	(3, 3, 3, 3)	(2, 2, 2, 2)
(1, 2, 2, 2)	(4, 1, 1, 1)	(3, 4, 4, 4)	(2, 3, 3, 3)
(2, 2, 3, 3)	(1, 1, 2, 2)	(4, 4, 1, 1)	(3, 3, 4, 4)
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- Each vector has the form $(q - 1, \dots, q - 1, q, \dots, q)$. First q doesn't change.
- Each arrow $v_j \rightarrow v_{j+1}$ represents $a_{v_j} \| b_{v_{j+1}}$.

It Works!

Lem. The arrows in C_t represent t^2 2-conflicted pairs.

Pf. Show that if a_{v_i} and b_{v_j} are incomparable, with $i + 1 \neq j$, then $a_{v_{j-1}} < b_{v_{i+1}}$.

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$$\begin{array}{ccc} & & \xrightarrow{\quad} \\ \begin{array}{c} q-2 \dots q-1 \\ q-1 \dots q \\ v_{j-1} \end{array} & \begin{array}{c} v_i \\ q-1 \dots q \\ q \dots q+1 \\ v_j \end{array} & \begin{array}{c} v_{i+1} \\ q \dots q+1 \\ q+1 \dots q+2 \end{array} \end{array}$$

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Now r is last index of q in v_{i+1} and s is first index of q in v_{j-1} . If overlap, then $s \leq r$. Now $r = s$, and $j = i + 1$. ■