ACO Comprehensive Exam March 2000 Name________________________

1. Graph Theory

(a) Construct a 3-regular graph on 32 vertices with no matching of size 15.

(b) Prove that every 3-regular graph on n vertices has a matching of size at least $7n/16$.

2. Probability

Let $X_1, X_2, \ldots$ be a sequence of independent random variables so that

$$X_n = \begin{cases} 2^n & \text{with probability } \dfrac{1}{n} \\ 0 & \text{with probability } 1 - \dfrac{1}{n} \end{cases}$$

(a) Does $X_n \xrightarrow{p} 0$?

(b) Does $\dfrac{X_1 + \cdots + X_n}{n} \xrightarrow{p} 0$?

(Recall that $\xrightarrow{p}$ refers to convergence in probability.) Explain your answers.

3. Linear Programming

We are given n points $p_1, \ldots, p_n$ on a straight line, where the distance of p_i from the origin is $a_i \geq 0$ and the weight of p_i is w_i, $i = 1, \ldots, n$. We want to locate a new point x on the straight line, such that the sum of the weighted distances of x from all the existing points is minimized.

(a) Formulate this problem as a linear program.

(b) Write the dual of the problem.

(c) Which positions for x correspond to basic solutions of this problem? (Hint: You may try to use surplus variables in the formulation.)

(d) For a given basic feasible solution of the primal, describe a corresponding basic dual solution. Give an interpretation for the dual variables.

(e) Suppose we want to use the simplex method to solve this problem. For a given basic feasible solution, what is an easy way to compute the reduced cost for each nonbasic variable? Which variable should enter the basis if the solution is not optimal? Give a physical interpretation for the simplex iteration.

(f) How would you approach this problem if the points $p_1, \ldots, p_n$ were located on the circumference of circle and we wanted to locate x on the circumference as well?

4. Discrete Optimization

Consider an integer program

$$\max \quad \sum_{i=1}^{n} c_i^2 x_i \quad \text{such that} \tag{1}$$

$$0 \le x_i \le 1 : i = 1, \ldots, n \tag{2}$$

$$\sum_{i=1}^{n} c_i x_i \le c_0 \tag{3}$$

$$x_i \text{ integer} : \; i = 1, \ldots, n \tag{4}$$

Here $n \ge 2$ is a positive integer, $c_0 > 0$ and $c_i \ge 0 : i = 1, \ldots, n$.

(a) Show that the greedy heuristic has limiting (as $n \to \infty$) worst-case performance $\frac{1}{2}$ or worse.

(b) Find the dimension of the LP relaxation.

(c) Find the dimension of the IP hull, i.e., of the convex hull of the feasible (integer) solutions.

(d) Find with proof necessary and sufficient conditions so that $x_i \ge 0$ is a facet of the IP hull.

(e) Find with proof necessary and sufficient conditions so that $x_i \le 1$ is a facet of the IP hull.

(f) Suppose that the c_i are i.i.d. uniformly distributed on $[0, T]$ for some $T \ge c_0$. Prove that the expected number of indices i, for which $x_i \le 1$ is a facet of the IP hull, converges to 1 as $n \to \infty$. (Hint: Find the probability that there are no facets.)

5. Algebra

For integers $n \geq 2$, let $\mathbb{Z}_n$ denote the ring of integers modulo n. Recall that $\mathbb{Z}_p$ is a field when p is prime. In this case the multiplicative group $\mathbb{Z}_p^*$ of units in $\mathbb{Z}_p$ is a *cyclic* group.

More generally, we denote

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid a \text{ is a unit in } \mathbb{Z}_n\}.$$

This group is not always cyclic.

Problem: Let p be an *odd* prime integer, and let g be a generator of the cyclic group $\mathbb{Z}_p^*$. Prove that either g or $g(p+1)$ is a cyclic generator of the group $\mathbb{Z}_{p^2}^*$, so that the group $\mathbb{Z}_{p^2}^*$ is also cyclic.

Remark: Do not confuse the ring $\mathbb{Z}_{p^2}$ (arithmetic modulo p^2) with the (very different) field $\mathbb{F}_{p^2}$ of order p^2.

6. Design and Analysis of Algorithms

You are given a set of n cities, and the costs of flights between each pair of cities. You wish to find the cheapest and most convenient route from city A to city B. Here cheapest means that the sum of the costs of the flights in the routing from A to B is as small as possible, and most convenient means that if there are more than one possible routings of minimum cost then you must choose the routing that minimizes the total number of flights. Give an efficient algorithm to solve this problem (including the running time and a proof of correctness).

7. Graph Algorithms

(a) Prove that a maximal matching is at least half the size of a maximum matching.

(b) Prove that if the Gomory-Hu tree for an edge weighted undirected graph G contains all $n-1$ distinct weights, then G can have only one minimum weight cut.