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1. Graph Theory

Use notation from Diestel, Graph Theory, Springer. All graphs are simple.

Lemma 3.2.1 in Diestel states: If a graph G is 3-connected and $|V(G)| > 4$, then G has an edge e such that G/e is also 3-connected.

(1) Use an argument similar to Lemma 3.2.1 to show: If a simple graph G is 3-connected, $f \in E(G)$, and $|V(G)| > 4$, then G has an edge $e \neq f$ such that G/e is also 3-connected.

(2) Use (1) to prove: If G is a 3-connected graph and $f \in E(G)$. Then G has an induced cycle C such that $f \in E(C)$ and $G - V(C)$ is connected.

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2. Probability

Problem Consider random variables Z, $X_1, X_2, \ldots$ $Y_1, Y_2, \ldots$ and $W_1, W_2, \ldots$ which are all independent of each other and with the following distributions.

$$\begin{array}{llllll} P(Z=1) & =1/2, & & P(Z=2) & =1/3, & \text{and } P(Z=3)=1/6, \\ P(X_i=1) & =.3 & \text{and} & P(X_i=0) & =.7 & \text{for each } i, \\ P(Y_i=1) & =.6 & \text{and} & P(Y_i=0) & =.4 & \text{for each } i, \text{ and} \\ P(W_i=1) & =.8 & \text{and} & P(Z_i=0) & =.2 & \text{for each } i. \end{array}$$

Finally let $U_k = X_k I_{\{Z=1\}} + Y_k I_{\{Z=2\}} + W_k I_{\{Z=3\}}$.

(a) Are the U_k's independent random variables?

(b) What is the variance of U_1?

(c) Does $\frac{1}{n}\sum_{i=1}^n U_i$ converge in probability and if so to which random variable?

(d) Does $\frac{1}{n}\sum_{i=1}^n U_i$ converge almost surely and if so to which random variable?

(e) Does $P\left(\frac{1}{\sqrt{n}}\left(\sum_{i=1}^n U_i - E[\sum_{i=1}^n U_i]\right) \in (3,4)\right)$ converge and to what?

3. Analysis of Algorithms

Given an undirected graph, $G = (V, E)$, $A \subseteq E$ is said to be an *edge-cover* if it covers all vertices. Give an efficient algorithm for finding a minimum cardinality edge-cover.

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4. Linear Programming

Part 1 (30 points) Read each of the following statements carefully to see whether is true or false. Justify your answers in a suitable way. (IF YOU ANSWER TRUE OR FALSE WITHOUT GIVING A JUSTIFICATION I WILL ASSUME THAT YOU HAVE GUESSED AND WILL CONSIDER THE ANSWER NULL.) In all these statements, problem (P) refers to the LP problem $\min\{c^T x : Ax = b,\ x \geq 0\}$ where A is a $m \times n$-matrix and $\mathcal{F}$ refers to the set of feasible solutions of this problem.

1) **(5 points)** The number of positive components of any extreme solution $\bar{x} \in \mathcal{F}$ does not exceed the rank of the matrix A.
2) **(5 points)** While solving the phase I problem associated with (P), if the unbounded criterion is satisfied, (P) must be unbounded.
3) **(5 points)** If (P) is feasible, termination occurs in solving the phase I problem associated with (P) only when all the artificial variables leave the basic vector.
4) **(5 points)** If (P) is unbounded then by changing the right hand side b, say to $\hat{b}$ where $\hat{b} \neq b$, then the new LP problem obtained may have an optimal solution.
5) **(5 points)** While solving (P) by the simplex algorithm, if the BFS in the beginning of a pivot step is degenerate, the objective value remains unchanged in this pivot step.
6) **(5 points)** A polyhedron of the form

$$\{(x, y) \in \Re^n \times \Re^p : A_1 x + A_2 y = b,\ x \geq 0,\ y \,\text{unrestricted}\,\},$$

where $A_1 \in \Re^{m\times n}$ and $A_2 \in \Re^{m\times p}$, always has an extreme point.

Part 2 (70 points) Let $M \in \Re$, $c \in \Re^n$, $b, d \in \Re^m$ and $A \in \Re^{m\times n}$ be given and consider the two LP problems:

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b,\ x \geq 0\} \\ (Q) & \min\{c^T x + M\xi : Ax + d\xi = b,\ x \geq 0,\ \xi \geq 0\}, \end{array}$$

where the variable in (P) is $x \in \Re^n$ and the variables in (Q) are $x \in \Re^n$ and $\xi \in \Re$. Assume that (P) has an optimal solution. Show that for M sufficiently large, say $M \geq M_0$, the LP problem (Q) has the following properties:

a) the optimal values of (P) and (Q) are the same;
b) if (x^*, ξ^*) is an optimal solution of (Q) then $\xi^* = 0$ and x^* is an optimal solution of (P);
c) if x^* is an optimal solution of (P) then $(x^*, 0)$ is an optimal solution of (Q).

5. Optimization

Use the max-flow/min-cut theorem to prove the following statements:

(a) Let P be an $n \times n$ bipartite matching problem (that is, a bipartite matching problem in which the bipartition divides the nodes into two sets of n nodes each). If every node has degree q, for any q such that $1 \leq q \leq n$, then P must have a perfect matching solution.

(b) Let $G = (V, E)$ be a simple, undirected graph on $|V| = n$ nodes. A traveling salesman problem on G can have a feasible solution only if every nonempty cut in G has size at least 2.

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6. Algebra

Prove or disprove (by giving an example) each of the following three statements:

1. The length and the factor groups of a Jordan-Holder series of a group determines the group uniquely.
2. Let G be a group and H a subgroup of G. Let $x \in G$ be such that $xHx^{-1} \subset H$. Then, $xHx^{-1} = H$.
3. Let f be a polynomial of degree n with coefficients in a field k. Let L be a splitting field of f over k. Then, $[L : k]$ divides $n!$.

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7. Approximation Algorithms

- Consider variants on the metric TSP problem in which the object is to find a simple path containing all the vertices of the graph. Three different problems arise, depending on the number (0, 1, or 2) of endpoints of the path that are specified. Obtain the following approximation algorithms.
- If 0 or 1 end points are specified, obtain a 3/2 factor algorithm.
- If both end points are specified, obtain a 5/3 factor algorithm.