

1. Graph Theory

Let $k, d \geq 1$ be integers, let G be a graph, and let $u, v \in V(G)$. Assume that for every set $X \subseteq V(G) - \{u, v\}$ with $|X| < k$ the vertices u and v are at distance exactly d in the graph $G \setminus X$. Prove that there exist paths P_1, P_2, \dots, P_k with ends u and v such that each of them has length exactly d and they are pairwise vertex-disjoint, except for their ends.

2. Probability

A coin is tossed repeatedly, heads occurring on each toss (independently) with probability p , ($0 < p < 1$). Find the probability generating function of the number T of tosses before a run of n heads has appeared for the first time. (For partial credit, solve for $n = 3$.)

3. Analysis of Algorithms

Let $G(U, V, E)$ be a bipartite graph, with $|U| = |V| = n$ and $E \subseteq U \times V$, as usual. For $X \subseteq U$, define the vertex neighborhood of X , $\Gamma(X)$, as the set of vertices in V that are incident to a vertex in X :

$$\Gamma(X) = \{v : (u, v) \in E \text{ for } u \in X, v \in V\}.$$

Let further $\gamma(G)$ be the minimum, over all $X \subseteq U$, of the ratio $|\Gamma(X)|/|X|$:

$$\gamma(G) = \min_{X \subseteq U} \frac{|\Gamma(X)|}{|X|}.$$

Give an algorithm that, on input $G(U, V, E)$, computes $\gamma(G)$ exactly, and has running time polynomial in n . You should give arguments for both correctness and running time. (Hint: γ “looks” like expansion, but it isn’t. What is the highest value that γ can attain?)

4. Linear Programming

1(a) Consider the standard linear programming problem

$$\begin{aligned} &\text{minimize} && cx \\ &\text{subject to} && Ax = b, \quad x \geq 0. \end{aligned} \tag{LP}$$

Describe the general primal-dual approach to solving this problem. In particular, describe the restricted primal problem that must be solved in each iteration.

(b) Consider the case where (LP) is the minimum cost assignment problem

$$\begin{aligned} \text{Minimize} & && \sum_{j=1}^n \sum_{i=1}^n c_{ij}x_{ij} \\ \text{Subject to} & && \sum_{i=1}^n x_{ij} = 1, \\ & && \sum_{j=1}^n x_{ij} = 1, \\ & && x_{ij} \geq 0 \quad \forall \quad i, j. \end{aligned}$$

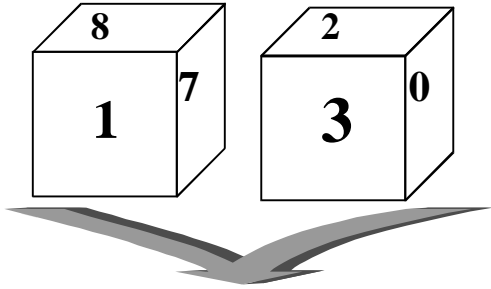
Show that the update of the dual variables in a primal-dual method for this problem requires one to find a minimum covering of edges by nodes in a bipartite graph.

(c) Assume the minimum cover in part (b) is available. Give an explicit formula for the dual variables corresponding to the restricted primal.

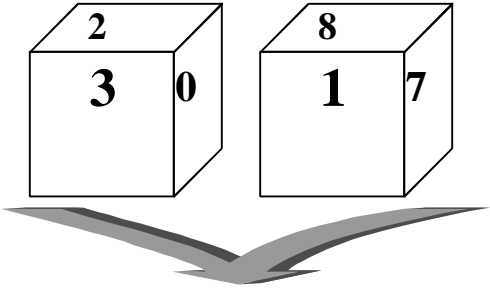
5. Combinatorial Optimization

Background

At home, I have a calendar that consists of two cubes. Each face of both cubes has a digit (0-9) on it, and with those cubes I can represent any day of the month. For example, on the 13th, I can place the first cube with the “1” in front to the left of the second cube with the “3” in front, to get 13. On the 31st, I can place the second cube (with the “3” in front) to the left of the first cube (with the “1” in front) to get 31.



Representation of "13"



Representation of "31"

Since my calendar can generate every date from 01 through 31 with two 6-face solids (cubes), it has a total “size” of $2 * 6 = 12$ faces. However, it has to “cheat” to achieve its 12-face solution; instead of having both a “6” and a “9”, it has a single face that can either be a “6” or a “9” subject to a 180-degree rotation. The first cube has faces $\{0, 1, 2, 3, 4, 5\}$ and the second cube has faces $\{0, 1, 2, 6, 7, 8\}$.

Question. In general, an instance of the *minimum-size 2-digit code generator problem (M2CG)* is specified by a set S of digits and a set C of 2-digit codes (s, t) where $s \in S$ and $t \in S$. A feasible solution to M2CG will be two sets of digits, D and E , such that for each $(s, t) \in C$, at least one of the following pairs of conditions is true:

- (i) $s \in D$ and $t \in E$,
- (ii) $s \in E$ and $t \in D$.

(In other words, for each code in C , one digit can be found in set D and the other can be found in set E .) An optimal solution to M2CG is a feasible solution that minimizes $|D| + |E|$.

Create a set-covering integer programming formulation using only $|S|$ variables, from which you can indirectly specify a solution to M2CG, and explain how to find the solution to M2CG given a solution to your integer program. (Hint: It may help to consider the graph $G = (S, C)$.)

6. Algebra

An algebraic integer is a complex number α which obeys a *monic* polynomial equation $P(\alpha) = 0$, where $P \in \mathbb{Z}[X]$. Let \mathbb{A} be the set of algebraic integers. Prove the following:

- (a) $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$, that is, any algebraic integer which is rational is an integer.
- (b) Eigenvalues of integer matrices are in \mathbb{A} and conversely, if $\alpha \in \mathbb{A}$, then it is an eigenvalue of an integer matrix.
- (c) \mathbb{A} is a ring.
- (d) Let $\alpha_1, \dots, \alpha_k \in \mathbb{A}$. Then there exists some m , a vector $v \in \mathbb{C}^m$, and k integer matrices B_1, \dots, B_k , such that $B_j v = \alpha_j v$, $1 \leq j \leq k$.
- (e) Let M be a matrix with elements in \mathbb{A} and α an eigenvalue of M . Then, $\alpha \in \mathbb{A}$.

7a. Computational Complexity Theory

Background:

- A language L is said to be in BPP iff there exists a polynomial-time recognizable binary relation R and a polynomial p such that

$$|\{r \in \{0, 1\}^{p(|x|)} : R(x, r) \neq \chi_L(x)\}| \leq \frac{2^{p(|x|)}}{3}.$$

(Here $\chi_L(x) = 1$ if $x \in L$ and 0 otherwise.)

- A language L is said to be in \mathcal{MA} iff there exists a polynomial-time recognizable ternary relation V and polynomials p, q such that:
 - If $x \in L$ then there exists $w \in \{0, 1\}^{q(|x|)}$ so that for every $r \in \{0, 1\}^{p(|x|)}$, $V(x, w, r) = 1$.
 - If $x \notin L$ then for every $w \in \{0, 1\}^{q(|x|)}$,

$$\text{Prob}(V(x, w, r) = 1) \leq \frac{1}{2}$$

where the probability is taken uniformly over all $r \in \{0, 1\}^{p(|x|)}$.

(That is, the class \mathcal{MA} is the class of languages accepted by public-coin interactive proof systems with *two* moves in which the first move is by the prover and the second move is by the verifier.)

- The following result about probability amplification for the class \mathcal{BPP} is known:

Amplification Theorem: A language L is in BPP iff there exists a polynomial-time recognizable binary relation R and a polynomial p such that

$$|\{r \in \{0, 1\}^{p(|x|)} : R(x, r) \neq \chi_L(x)\}| < 2^{\frac{p(|x|)}{3}}.$$

Question: Using the amplification theorem, show that $BPP \subseteq \mathcal{MA}$.

7b. Approximation Algorithms

Consider the minimum makespan scheduling problem: Given processing times for n jobs and an integer m , find an assignment of the jobs on m identical machines so as to minimize the completion time.

Consider the following algorithm for this problem: sort jobs by decreasing times. Schedule jobs in this order, always assigning the next job to the machine that has least total work assigned to it. Show that this algorithm achieves a factor of $4/3$. Provide a tight example.

7c. Randomized Algorithms

In the network reliability problem, we are given a graph where each edge fails with some probability p . We are interested in estimating the probability $\text{FAIL}(p)$ with which the network gets disconnected. A simple Monte Carlo estimator for this probability is based on repeated trials as follows: Randomly delete each edge in the graph independently with probability p . Then check to see if the graph becomes disconnected.

- a) Explain how this can be used to estimate $\text{FAIL}(p)$ in the case that this probability is not too small.
- b) Using Chernoff bounds, determine how many trials suffice (up to big- O notation) to estimate $\text{FAIL}(p)$ to within a factor $(1 \pm \epsilon)$ with probability at least $1 - n^{-1}$.