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1. Graph Theory

Let G be a graph such that every edge of G belongs to some perfect matching of G. Let $X, Y \subseteq V(G)$ be disjoint, let M be a perfect matching in $G \backslash (X \cup Y)$, and assume that there exists a path in G with one end in X and the other end in Y. Prove that there exists an M-alternating path in G with one end in X and the other end in Y.

2. Probability

Consider a symmetric random walk with an absorbing barrier at N and a reflecting barrier at 0 (so that when the particle is at 0, it moves to 1 in the next step). Let $p_k(j)$ be the probability that the particle having started at k, visits 0 exactly j times before being absorbed at N; we make the convention that if $k = 0$, then the starting point counts as one visit.

Show that $p_k(j) = (N - k)/N^2$ times $(1 - 1/N)^{j-1}$, $j \geq 1$, $0 \leq k \leq N$.

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3. Analysis of Algorithms

Suppose that a graph G has a minimum spanning tree already computed. How quickly can the minimum spanning tree be updated if a new vertex and incident edges are added to G?

4. Linear Programming

Consider the linear program in standard form: minimize cx subject to $x \geq 0$ and $Ax = b$, with dual problem $\pi A \leq c$, maximize πb.

a. Show that for those c for which the dual is feasible, the recession cone of the dual polyhedron is the same regardless of c.

b. What is the recession cone of the dual polyhedron when the primal problem is a set packing problem (A is an $m \times n$ 0-1 matrix containing an $m \times m$ identity matrix and b is a column vector of 1's)?

c. Show that the constraints on the region of b's such that the primal is feasible are precisely $rb \leq 0$ for all of the rays r of the recession cone of the dual polyhedron. Use question b. to give an example.

5. Combinatorial Optimization

Consider the (undirected) TSP on graph $G = (V, E)$ with edge costs c, node-arc incidence matrix A. For $S \subset V$ let $\delta(S)$ denote the set of edges in E with exactly one endpoint in S.

(i) Prove that the following LP relaxation of the TSP can be solved in polynomial time.

$$\min \sum_{e \in E} c_e x_e \quad \text{s.t.}$$

$$\begin{aligned} Ax &= 2; \\ \sum_{e \in \delta(S)} x_e &\geq 2 : 0 < |S| < |V| \\ 0 &\leq x \leq 1 \end{aligned}$$

(ii) Let $\pi \in \Re^{|V|}$ be a vector of vertex weights, and define the corresponding modified edge costs $f_{ij} = c_{ij} - \pi_i - \pi_j$. Let $w \in V$ such that $\delta(\{w\}) \geq 2$. Define $\alpha = 2 \sum_{v \in V} \pi_v$. Let β denote the minimum over all pairs of distinct edges $e_1, e_2 \in \delta(\{w\})$ of $f_{e_1} + f_{e_2}$. Let γ denote the cost of a minimum spanning tree on $G = (V - w, f)$ (where f is understood to not include edges incident on w.)

Show that $\alpha + \beta + \gamma$ is a valid lower bound for the TSP.

(iii) Show that $\alpha + \beta + \gamma$ is not a stronger lower bound than the value of the LP relaxation of part (i).

6. Algebra

Let G be a transitive group of permutations of a set X and H the stabilizer of an element $x \in X$.

1. Show the equivalence of the following properties:

   (a) Every subgroup H' of G containing H is equal to H or G.

   (b) Every subset Y of X such that, for all $g \in G$, gY is either contained in Y or disjoint from Y, is equal to X or consists of a single element.

   A transitive permutation group G satisfying the above properties and not consisting only of the identity element is called *primitive*.

2. Let G be a transitive group of permutations of a set X; suppose G is primitive. Show that every normal subgroup of G distinct from $\{e\}$ is transitive.

7. Approximation Algorithms

Consider the following problem:

(Feedback vertex set) Given an undirected graph $G = (V, E)$ and a function w assigning nonnegative weights to its vertices, find a minimum weight subset of V whose removal leaves an acyclic graph.

A natural greedy algorithm for finding a minimum feedback vertex set is to repeatedly pick and remove the most cost-effective vertex, i.e., a vertex minimizing $w(v)/\delta_H(v)$, where H is the current graph, until there are no more cycles left. Give examples to show that this is not a constant factor algorithm. What is the approximation guarantee of this algorithm?