ACO Comprehensive Exam October 2004 Name ______________________________

1. Graph Theory

Let $k > 1$ be an integer. Prove that every k-connected graph on at least $2k$ vertices has a cycle of length at least $2k$.

ACO Comprehensive Exam   October 2004   Name ______________________________

2. Probability

Let us consider d balls numbered from 1 to d (for $d > 1$), divided between two urns A and B. A number i is randomly chosen between 1 and d and then one of the urns is chosen at random, into which the ball with the number i is placed. All the draws are assumed to be independent and uniform. Let X_n denote the number of balls present in urn A after n draws.

(a) Determine the transition matrix of the Markov chain $\{X_n\}$, $n \geq 0$. Is it irreducible? Is it aperiodic?

(b) Determine its stationary distribution.

3. Analysis of Algorithms

Let $T(V, E)$ be a tree given in its adjacency list representation. For vertices $u, v \in V$, let $dist(u, v)$ denote the length of the path from u to v in T. Give an $O(|V|)$ time algorithm that determines the value $\max_{u,v \in V} dist(u, v)$.

ACO Comprehensive Exam October 2004 Name ____________________

4. Linear Programming

SIMPLEX ALGORITHM: 5 points The standard version of Phase I of the simplex method for the system

$$Ax \geq b; x \geq 0$$

where $b \geq 0$ solves

$$\begin{array}{rcl} & \min & \mathbf{1} \cdot w \quad s.t. \\ Ax - Iy + Iw & = & b \\ x, y, w & \geq & 0 \end{array}$$

starting from initial basis w.

A 2nd version employs a single artificial variable w_0. It solves

$$\begin{array}{rcl} & \min & w_0 \quad s.t. \\ Ax - Iy + \mathbf{1}w_0 & = & b \\ x, y & \geq & 0 \\ w_0 & \geq & 0. \end{array}$$

1. **1 pt** For the 2nd version, how would you select an initial basis and what is the corresponding basic feasible solution?
2. **2 pts** Compare the reduced costs of the x and y variables in the standard version with the reduced costs of these variables in the 2nd version (at the first iteration only).
3. **2 pts** Compare the computational properties and behaviors of the two versions.

DUALITY: 5 points One form of Farkas's Lemma states that either the system

$$Ax \leq b; x \geq 0$$

has a solution, OR the system

$$\begin{array}{rcl} \pi^T A & \geq & 0 \\ \pi^T b & < & 0 \\ \pi & \geq & 0 \end{array}$$

has a solution, but not both.

1. **1 pt** Derive from the above a Farkas type lemma regarding the system

$$Ax = b; x \geq 0.$$

2. **2 pts** Use your lemma to prove that the set $\mathcal{D}$ of all nonnegative linear combinations of the vector set $\{d^i : i \in I\}$, where $d^i \in \mathbb{R}^n$, is closed if I is finite.
3. **2 pts** Prove directly that $\mathcal{D}$ is closed by showing that the convex hull of $\{d^i : i \in I\}$ is closed and applying an appropriate continuity argument.

ACO Comprehensive Exam   October 2004   Name ______________________

5. Combinatorial Optimization

Note: You may use well known facts about network flows without proof.

Consider the following arc covering problem (ACP): Given a complete directed Euclidean graph $D = (N, A)$ with node set N, arc set A, arc costs $c(a)$ for $a \in A$, and a special arc set $L \subseteq A$, find a minimum cost set of simple cycles covering the arcs in L (a cycle cover of L).

1. Give a polynomial time optimal algorithm for solving ACP. Determine the complexity of your algorithm.
2. Now consider a constrained version of ACP, the cardinality constrained arc covering problem (CCACP): Given a complete directed Euclidean graph $D = (N, A)$ with node set N, arc set A, arc costs $c(a)$ for $a \in A$, and a special arc set $L \subseteq A$, find a minimum cost cycle cover of L, such that the cardinality of each cycle, i.e., the number of arcs in the cycle, is less than or equal to a pre-specified number K. The addition of this constraint makes the optimization problem NP-hard.
   (a) Give a heuristic with a constant approximation bound (i.e., the value of the heuristic solution must be guaranteed to be less than or equal to a constant times the value of the optimal solution). Is your approximation bound tight? Why?
   (b) Give an exact method for solving the CCACP.

6. Algebra

Let $\mathbb{C}$ denote the field of complex numbers. Determine all polynomials $f(x) \in \mathbb{C}[x]$ of degree 2004 such that

- $f(x)$ is monic,
- it has distinct roots, and
- the product of any two roots of $f(x)$ is also a root of $f(x)$.

7. Computational Complexity

Define the complexity class $\oplus P$ as follows: A language $L \in \oplus P$ if there is a nondeterministic Turing machine M such that for all strings x, we have $x \in L$ if and only if the number of accepting computations of M on input x is odd.

(This definition is from the book "Computational Complexity" by Christos H. Papadimitriou, page 448.)

Two undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $\sigma : V_1 \to V_2$ such that for every pair of vertices $i, j \in V_1$, $(i, j) \in E_1$ if and only if $(\sigma(i), \sigma(j)) \in E_2$.

Define the problem **POLY-GI**: Given as input two graphs G, H on n vertices such that they have at most $p(n)$ isomorphisms, where $p(n)$ is a polynomial in n, decide if G and H are isomorphic.

(The **POLY-GI** problem is to decide if the graphs are isomorphic given that they have at most $p(n)$ isomorphisms. That is, the decision algorithm can assume that the graphs have at most $p(n)$ isomorphisms and does not have to check this property.)

Question: Show that **POLY-GI** is in $\oplus P$.

7. Randomized Algorithms

Here is a way to construct a random permutation π on $[1, n]$ given a black box that outputs number independently and uniformly at random from $[1, k]$ where $k \geq n$. If we compute a function $f : [1, n] \rightarrow [1, k]$ with $f(i) \neq f(j)$ for $i \neq j$, this yields a permutation; simply output the numbers $[1, n]$ according to the order of the $f(i)$ values. To construct such a function f, do the following for $j = 1, ..., n$: choose $f(j)$ by repeatedly obtaining numbers from the black box and setting $f(j)$ to the first number found such that $f(j) \neq f(i)$ for $i < j$.

a) Prove that this approach gives a permutation chosen uniformly at random from all permutations.

b) Find the expected number of calls to the black box that are needed when $k = n$ and $k = 2n$.

c) For the case $k = 2n$, argue that the probability that each call to the black box assigns a value to $f(j)$ to some j is at least $1/2$. Based on this, use a Chernoff bound to bound the probability that the number of calls to the black box is at least $4n$.