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1. Graph Theory

Let $\mathcal{I}$ be a set of 2005 intervals on the real line such that for every subset $\mathcal{I}' \subseteq \mathcal{I}$ of size 11 there exist three distinct intervals $I, J, K \in \mathcal{I}'$ such that $I \cap J \cap K \neq \emptyset$. Prove that there exists a real number that belongs to at least 401 of the intervals in $\mathcal{I}$.

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2. Probability

Find $\Pr(1 < X < 4)$ if

(i) $X = X_1 + X_2$

(ii) $X = \min(X_1, X_2)$

in which X_1 and X_2 are independent, identically distributed random variables possessing the probability law as described below:

(a) X_1 and X_2 are Poisson, with mean 2.

(b) X_1 and X_2 are geometric, with mean 2.

3. Analysis of Algorithms

For a directed graph $G = (V, E)$, we are interested in the following property: for every pair of distinct vertices u and v, there is either a path from u to v or a path from v to u. Given G in adjacency list representation, give an $O(|V| + |E|)$ time algorithm to determine if G satisfies the above property. Explain your algorithm in words, do not give code, and prove correctness of your algorithm.

4. Linear Programming

(a) Consider a primal-dual pair of linear programs

$$
\begin{aligned}
(P) \quad & \min_x \{c^T x : Ax \geq b\} \\
(D) \quad & \max_y \{b^T y : A^T y = c, \ y \geq 0\}
\end{aligned}
$$

Prove that if both problems are feasible, then the feasible set of at least one of them is unbounded.

(b) Prove that polyhedron P in $\mathbf{R}^n$ given by a list of m inequalities of the form $a_i^T x \leq 1$, $i = 1, \ldots, m$, is bounded if and only if every vector from $\mathbf{R}^n$ can be represented as a linear combination of a_i with nonegative coefficients, and even as a combination of this type in which at most n coefficients are positive.

5. Integer and Combinatorial Optimization

1. (4 points) An integer program $\max b \cdot x : Ax \geq d; x \in B^n$ is to be solved by branch and cut. As a first step, the LP relaxation is solved.

   - (2 points) Suppose that the LP relaxation is solved, yielding an integer optimum solution. Valid cuts are then added to the LP formulation, and the tighter LP relaxation is solved from scratch. Prove or disprove: the tighter formulation must yield an integer optimum solution.
   - (2 points) Suppose that the objective coefficients c are generated at random as follows: $c_i = y_i + y_{i+1}$, where $y_1 \dots y_{n+1}$ are independent Bernoulli variables each with $p = 1/2$. Note that the c_i are not jointly independent. Prove or disprove: if valid cuts are added to the LP formulation, the probability that the LP relaxation has an optimal integer solution cannot decrease.

2. (4 points)

   In an old house in Paris that was covered with vines
   Lived 12 little girls in two straight lines.
   The smallest one was Madeline.
   Ludwig Bemelmans

   Each girl is a different size and wears a different color shoe. The lines are parallel, forming 6 rows of two from front to back. Cam and the red-shoed girl are in adjacent rows. The biggest girl is ahead of Rhee (in the same line). The blue-shoed girl is smaller than everyone in the left line. Eft and Alice are in the same line, which is different from Madeline's. Formulate the problem of determining the girls' locations, shoe colors, and (ordinal) sizes as an integer program.

3. (2 points) Find the break-even point between the following choices as the core data structure for Dijkstra's shortest path algorithm, with respect to average vertex degree: (i) a priority queue implemented by binary heaps; (ii) a vertex-vertex matrix (non-sparse representation). The algorithm evaluation criterion is worst-case performance.

6. Algebra

Suppose σ and τ are chosen uniformly at random from S_6 (the symmetric group on 6 letters). What is the probability that σ and τ commute?

7. Computational Complexity

7. Randomized Algorithms

In a school with $2n$ students, we want to randomly partition the students into two classes of approximately equal size. Suppose each student independently chooses one of the two classes, with each class being equally likely. The expected number of students in each class is then n. We now want to explore how big their difference is likely to be. Let X_1 and X_2 denote the number of students in each of the two classes. Prove that for any $\epsilon > 0$ there is a constant $c > 0$ such that the probability $Pr\left[X_1 - X_2 > c\sqrt{n}\right] \leq \epsilon$.

7. Graph Algorithms

(a) Let $G = (V, E)$ be an undirected connected graph with non-negative weights on edges, and let T be a spanning tree in G. Define the *bottleneck edge in* T to be the heaviest edge in it. A *bottleneck spanning tree* is a spanning tree whose bottleneck edge is minimum possible.

Give an efficient algorithm for finding a bottleneck spanning tree in G. You are allowed to use an MST algorithm as a subroutine. (You will be graded on the efficiency of your algorithm.)

(b) A permutation matrix is a 0/1 matrix that has all row sums and column sums of 1. Prove the Birkhoff-von Neumann Theorem: Any doubly stochastic matrix, A, can be written as a convex combination of permutation matrices, i.e.,

$$A = c_1 P_1 + \cdots + c_k P_k,$$

where $\sum_{i=1}^{k} c_i = 1$, each $c_i \geq 0$ and $P_1, \cdots P_k$ are permutation matrices.

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7. Approximation Algorithms