ACO Comprehensive Exam **March 2005** **Name** ____________________________

1. Graph Theory

Prove that there exists a constant K such that every graph G has either two disjoint cycles, or a set X of at most K vertices such that $G \backslash X$ has no cycles. To receive full credit you must show this for $K = 3$. Give an example showing that the above fails for $K = 2$.

2. Probability

A. Ten percent of the surface of a sphere is colored blue while the rest is colored red.

Show that irrespective of the manner in which the colors are distributed, it is possible to inscribe, in that sphere, a cube with all its vertices red.

B. Each member of a group of n players roll a fair die. For any pair of players who throw the same number, the group scores one point.

Find the mean and variance of the total score of the group.

3. Analysis of Algorithms

There are N locations (points) in a city. The distance between the points are given by an N by N matrix $\{d_{ij}\}$, where d_{ij} is the distance between points i and j. There are k taxicabs in this city, all of which start out at a particular location z. A sequence of n requests are to be satisfied by the taxicabs. The ith request consists of a starting point s_i and a destination point t_i. To satisfy a request, one of the taxicabs drives from wherever it is to s_i, picks up the passenger, then drives from s_i to t_i and lets out the passenger, and waits there until it is ordered to satisfy another request. The requests must be processed in their order of occurrence, and a taxicab may only move to get to a request it is about to satisfy, or from the start to the destination point of a request. Give an algorithm that takes the complete sequence of requests, the distance matrix, and the number of taxicabs k, and find a solution that minimizes the total distance moved by all taxicabs. The algorithm should have running time polynomial in k, n, and N. Explain why your algorithm works.

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4. Linear Programming

FORMULATION: 2 points We are given $A \in \Re^{m \times n}$, $b \in \Re^m$, $\pi \in \Re^n$, $\pi_0 \in \Re$. Formulate a linear program which determines whether $\pi^T x \leq \pi_0$ is CG rank 0 with respect to $Ax \leq b$, i.e., whether

$$\{x : Ax \leq b\} \subseteq \left\{x : \pi^T x \leq \pi_0\right\}.$$

POLYHEDRA: 3 points Let $P^i : i = 1, 2, \ldots t$ be polyhedra in $\Re^n$. For any polyhedra P, S and $\alpha \in \Re$ let $\alpha P \equiv \{\alpha x : x \in P\}$, and let $P + S$ denote the set sum $\{p + s : p \in P, s \in S\}$. Now let

$$Q = \sum_{i=1}^{t} (-1)^{i+1} i P^i.$$

Prove or disprove: Q is a polyhedron.

DUALITY: 5 points The Farkas Lemma states that exactly one of the two systems has a solution:

$$\begin{aligned} Ax &= b; \quad x \geq 0 \\ \pi^T A &\geq 0; \quad \pi^T b < 0. \end{aligned}$$

1pt Derive the alternative system for

$$Ax \leq b$$

from the given Farkas Lemma.

4pts Prove the correctness of your alternative system by Fourier-Motzkin elimination. Hint: show that FM elimination corresponds to multiplication by a nonnegative matrix (partial credit for either showing this is true, or for completing the proof assuming this is true).

5. Integer and Combinatorial Optimization

3 points We are given $A \in \Re^{m\times n}$, $b \in \Re^m$, $\pi \in \mathcal{Z}^n$, $\pi_0 \in Z$.

Formulate an integer linear program which determines whether or not $\pi^T x \leq \pi_0$ is CG rank 1 with respect to $Ax \leq b$.

7 points Let $P \subset \Re^{\binom{2n}{2}}$ denote the polyhedron

$$\{x \mid 0 \leq x \leq 1; Ax \leq 1\},$$

and let P' denote polyhedron

$$\{x \mid 0 \leq x \leq 1; Ax = 1\},$$

where A is the node-arc incidence matrix of K_{2n}. Let

$$Q = conv\left\{Z^{\binom{2n}{2}} \cap P\right\},$$

the convex hull of integer points of P.

(4pts) Determine, with proof, the dimensions of P, P', and Q. Your proofs should be self-contained.

(2pts) Determine, with proof, whether or not $x_1 \leq 1$ is a facet of Q.

(1pts) Define Q' in the analogous way from P'. How many facets does Q' have?

6. Algebra

1. Prove that a finite group with three or more elements has at least one non-trivial automorphism.

2. Let $R \subseteq S$ be commutative integral domains such that every element of S is the root of a monic polynomial with coefficients in R. Prove that R is a field if and only if S is a field.

7. Computational Complexity

1. Consider non-constant levels of the polynomial hierarchy defined as follows: A language L is in $\Sigma^P_{a(n)}$ iff there is an alternating Turing machine M such that: (a) M starts in an existential state and alternates at most $O(a(n))$ times, (b) M runs in polynomial time, and (c) M accepts L.

   Show that $NSPACE(\log n)$ is *properly* contained in $\Sigma^P_{\log^2 n}$.

7. Randomized Algorithms

a) In an election with two candidates using paper ballots, each vote is misrecorded with probability $p = 0.02$. Use Chernoff bounds to bound the probability that more than 4% of the votes are misrecorded in an election of $1,000,000$ ballots.

b) Assume that the misrecorded ballot always counts as a vote for the other candidate. Suppose that candidate A received 510,000 votes and that candidate B received 490,000 votes. Use Chernoff bounds to bound the probability that candidate B wins the election owing to misrecorded ballots. Specifically, let X be the number of votes for candidate B that are misrecorded and let Y be the number of votes for candidate B that are misrecorded. You should bound $\Pr((X > k) \cup (Y < \ell))$ for suitable choices of k and ℓ.

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7. Graph Algorithms

- Prove that a bipartite graph with maximum degree Δ has an edge coloring using exactly Δ colors.
- Give a polynomial time algorithm for finding a minimum $s-t$ cut with the least number of edges crossing from the s side to the t side.

7. Approximation Algorithms

- Consider the following variant of the set multicover problem in which the cost of picking an item a multiple number of times does not grow linearly, but instead grows according to a concave function. For each set S_i we are given a concave function f_i specifying the cost of picking this set multiple times. The problem again is to satisfy all coverage requirements of elements at minimum cost. Give a factor H_n algorithm for this problem.

- Consider the following generalization of the maximum cut problem.

  Linear equations over GF[2]: Given m equations over n GF[2] variables, find an assignment for the variables that maximizes the number of satisfied equations.

  1. Show that if $m \leq n$, this problem is polynomial time solvable.
  2. In general, the problem is NP-hard. Give a factor $1/2$ randomized algorithm for it, and derandomize using the method of conditional expectation.