1. Graph Theory

Let m be a positive integer and let G be a $2m$-colorable graph. Show that G contains a bipartite subgraph H such that $|E(H)| \geq \frac{m}{2m-1}|E(G)|$.

2. Probability

With the help of the Strong Law of Large Numbers, find

$$\lim_{n\to+\infty} \int_0^1 \cdots \int_0^1 f\left(\frac{x_1+\cdots+x_n}{n}\right) dx_1 \cdots dx_n,$$

where f is a continuous and bounded function from $\mathbb{R}$ to $\mathbb{R}$.

3. Analysis of Algorithms

A (d, c, α)-expander is a graph $G = (V, E)$ where each node has degree at most d, and every subset $S \subseteq V$ with at most cn nodes has $|N(S)| \geq \alpha|S|$, where $N(S)$ is the set of vertices in V that have a neighbor in S.

Starting with the set $V = \{1, 2, \ldots, n\}$ of nodes, add a random matching between the vertices as follows: (a) choose a permutation $v_1, v_2, ..., v_n$ of the nodes uniformly at random, and (b) add the edges (i, v_i) for all i. (We may have parallel edges and self-loops; that is fine.) Repeat this process $d = 600$ times; let G denote the resulting graph. Prove that G is a $(2d, 7/20, 3/2)$-expander with probability at least $1/2$.

4. Linear Programming

A discrete random variable ξ with zero mean takes values from the set $\{i/2005,\ i = 0, \pm 1, \pm 2, \ldots, \pm 2005\}$. What can be the largest possible expectation of $\exp\{2\xi\}$?

5. Combinatorial Optimization

In a round-robin competition with n players, where each player has a match with each other player, let w_i denote the number wins for player i (we assume there are no ties in the matches). Describe a network optimization model that gives a polynomial-time algorithm to test if a given vector $w = (w_1, w_2, \ldots, w_n)$ arises from such a competition.

6. Algebra

If G is a group of order 140 and H is a subgroup of G of order 35, prove that H is normal in G.

7. Graph Algorithms

(1) Let $G = (V, E)$ be an undirected graph with positive integral weights on edges, and let $s \in V$ be a special vertex. Let T be the shortest path tree obtained by running Dijkstra's algorithm from s in G. Is T a minimum spanning tree in G? If yes, prove it; otherwise, give a counterexample.

(2) Define a bottleneck spanning tree in an undirected graph to be one that minimizes the maximum weight edge used. Prove that any minimum spanning tree is a bottleneck spanning tree. Is the converse not true? Give a proof or counterexample.

7. Randomized Algorithms

There is a driver who visits n cities $c_1, \ldots, c_n$ in order, and then repeats. Thus, the driver does the following cycle:

$$c_1 \to c_2 \to c_3 \to \cdots \to c_{n-1} \to c_n \to c_1 \to c_2 \cdots \to c_n \to c_1 \to c_2 \ldots$$

The driver has n packages initially. At each visit to a city, he drops a single package with probability p (independently between visits), and with probability $1-p$ does not drop anything. The driver stops once all n packages are dropped.

(a) Suppose $p > 1/2$. Give a high probability upper bound on the maximum number of packages at any city.

(b) Suppose $p < 1/n^6$. Once again, give a high probability upper bound on the maximum number of packages at any city.

To be precise, say high probability means $\geq 1-n^{-3}$ and state your bound for n sufficiently large. For answers that depend on n (e.g., $\sqrt{n}$), it's OK to get it within a constant factor (so $O(\sqrt{n})$ is sufficient).