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1. Graph Theory

Let G be a graph which is both triangle-free and k-connected, ($k \geq 2$).

(a) Let S be a k-cut in G and C a component of $G - S$. Show that if S is not independent in G then $|C| \geq k$.

(b) Give an example of G such that for any k-cut S in G, $G - S$ has a component of size less than k.

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2. Probability

When the sprows burst in a laptometer, we are faced with the failures of the first kind. Now, every sprow that bursts causes failures of the second kind (independently of the number of failures of the first kind and of the other sprows). Suppose that the number of failures of the first kind during one hour follows the $\mathrm{Po}(\lambda)$-distribution and the number of failures of the second kind caused by one sprow follows the $\mathrm{Bin}(n, p)$-distribution. Find the mean and variance of the total number of failures during one hour.

3. Analysis of Algorithms

You are given an unsorted array of n distinct items from some totally ordered universe (say, the integers). It is not possible for you to compare two items directly. Instead, you only have access to a **Median** oracle that tells you, in order $O(1)$ time, which of three arbitrary items chosen from the array is between the other two.

- Prove that it is impossible to find the minimum element of the array using only the **Median** oracle.
- Describe and analyze an efficient algorithm for finding the minimum and maximum elements of the array using the **Median** oracle.
- Describe and analyze an efficient randomized algorithm to sort the array using the **Median** oracle, in either increasing or decreasing order.
- Prove that your sorting algorithm is optimal.

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4. Linear Programming

A. Solve the linear programming program

$$\min_x \left\{ \sum_{j=1}^{2005} j^2 x_j : 0 \le x_j \le 1, \sum_j x_j = 999 \right\}.$$

B. The objective c in a linear programming program

$$\mathrm{Opt}(c) = \min_x \left\{ c^T x : Ax \ge a \right\}$$

with nonempty and bounded feasible set can vary in a polyhedral set given by $Bc \le b$. Represent the worst-case optimal value

$$\mathrm{Opt}_* = \sup_{c: Bc \le b} \mathrm{Opt}(c)$$

as the optimal value in an explicit linear programming program.

C. Let $s_k(x)$ be the sum of k smallest coordinates in a vector $x \in \mathbf{R}^n$ (e.g., $s_3((1,1,3,2)^T) = 1+1+2 = 4$). Reformulate the optimization problem

$$\max_x \{ s_k(x) : Bx \le b \}$$

with $m \times n$ matrix B as a linear programming program with polynomial in m, n number of variables and constraints.

5. Combinatorial Optimization

Let $G = (V, E)$ be the complete graph on the vertex set V, let $r \in V$ be a specified root vertex, and let $(c_e : e \in E)$ be specified edge costs. For a fixed integer k, consider the problem of finding a family $\mathcal{C}$ of circuits such that $|\mathcal{C}| = k$, vertex r is contained in each circuit, and every vertex in $V - \{r\}$ is contained in exactly one circuit. Furthermore, suppose we have a family B of subsets of $V - \{r\}$ such that for each $B \in \mathcal{B}$ it is not permitted to have a circuit $C \in \mathcal{C}$ such that the vertices B appear consecutively in C.

(a) Formulate as an IP the optimization problem of finding such a family $\mathcal{C}$ having minimum total edge cost.

(b) Suppose that we have a polynomial-time algorithm to test if a set of vertices is a member of $\mathcal{B}$. Show that the LP relaxation of the formulation given in part (a) can be solved in polynomial time. (Note that $\mathcal{B}$ may have exponentially many members.)

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6. Algebra

Let p be a prime number, let $\mathbf{F}_p$ be the field with p elements, and let $G = \mathrm{GL}_2(\mathbf{F}_p)$ denote the group of 2×2 invertible matrices over $\mathbf{F}_p$.

(a) Show that G has order $p(p-1)^2(p+1)$.

(b) Show that every element of G has order dividing either $p(p-1)$ or $p^2 - 1$.

7. Randomized Algorithms

Consider the following scheme for shuffling n distinct cards. A deck is a permutation of the cards. At each step, we choose a random card and place the chosen card at the top of the deck (i.e., the card is moved to the beginning of the permutation). Let X_0 denote the initial permutation, and let X_t denote the permutation after t steps.

a) Given that the uniform distribution over permutations of the n cards is a stationary distribution of the process, argue that it is the *unique* stationary distribution.

b) Suppose that the process is run for k steps and k distinct cards were chosen in these k steps. Describe the distribution over permutations for X_k.

c) Let T denote the time until the process is within variation distance $\leq 1/4$ of the uniform distribution. Upper bound T. (We are expecting the right asymptotic bound, and are not concerned about constant factors.)

7. Graph Algorithms

Let D be a directed graph, and let G be its underlying undirected graph. A set $\mathcal{C}$ of directed cycles in D is called a *directed cycle basis of* D if for every (undirected) cycle C in G there exist cycles $C_1, C_2, \ldots, C_k \in \mathcal{C}$ such that $E(C) = E(C_1) \triangle E(C_2) \triangle \cdots \triangle E(C_k)$, where $\triangle$ denotes symmetric difference. Design a polynomial-time algorithm that given a digraph D outputs either a directed cycle basis in D, or a valid statement that D has no directed cycle basis. Prove correctness of your algorithm and prove that it runs in polynomial time.

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7. Approximation Algorithms

1. Let MAX-LIN-2 be the following problem: Given n boolean variables $\{x_1, x_2, \ldots, x_n\}$ and a set of m linear equations (modulo 2) over these variables, find an assignment to the variables that satisfies maximum number of equations. Note that every equation is of the form:

$$x_{i_1} \oplus x_{i_2} \ldots \oplus x_{i_k} = b \qquad 1 \leq k \leq n,\ b \in \{0, 1\}$$

   - Show that the problem is NP-hard by giving a reduction from MAX-CUT.
   - Give a randomized polynomial time algorithm that achieves a 2-approximation.
   - Derandomize your algorithm (Hint: use the method of conditional expectations).

2. Let MINIMUM-LINEAR-ARRANGEMENT (MLA) be the following problem: Given an undirected graph $G(V, E)$, $|V| = n$, find a numbering of the vertices from 1 to n, i.e. a bijection $h : V \mapsto \{1, 2, \ldots, n\}$ so as to minimize

$$\sum_{(u,v)\in E} |h(u) - h(v)|$$

Show that the following is an LP-relaxation of MLA. The LP has a variable $d(u, v)$ for *every* vertex pair (u, v) which is thought of as distance between the two vertices. This distance function is supposed to be a metric.

$$\begin{aligned} \text{minimize} \sum_{(u,v)\in E} d(u,v) \\ d(u,v) \geq 0\ \forall u, v \in V \\ d(u,v) + d(v,w) \geq d(u,w)\ \forall u, v, w \in V \\ \sum_{u\in S} d(u,v) \geq \frac{1}{4}(|S|^2 - 1) \quad \forall S \subseteq V, v \in S \end{aligned}$$

Show that the LP can be solved in polynomial time by giving a polynomial time separation oracle for it.

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7. Computational Complexity

1. (a) Show that every language in $DSPACE(O(n^2))$ reduces in polynomial time to a language in $DSPACE(O(n))$.

   (b) Show that $\mathcal{NP} \neq DSPACE(O(n))$.

2. Two undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $\sigma : V_1 \rightarrow V_2$ such that for every pair of vertices $i, j \in V_1$, $(i, j) \in E_1$ iff $(\sigma(i), \sigma(j)) \in E_2$. The *Graph Isomorphism* problem is to decide if two input graphs are isomorphic.

   Show that if the *Graph Isomorphism* problem is in $\mathcal{BPP}$ then it is in $\mathcal{RP}$. (Note: The $\mathcal{RP}$ algorithm should not make any errors when the input graphs are not isomorphic.)