ACO Comprehensive Exam March 1999 Name____________________________

1. Graph Theory

Let n be a positive integer, and let X and Y be two disjoint sets of size n. Consider the probability space of all simple bipartite graphs with bipartition (X, Y), where $x \in X$ and $y \in Y$ are adjacent with probability p, independently of other pairs of vertices. Prove that if $p = (5 \log n)/n$, then almost every graph in this model has a perfect matching.

2. Probability

a. Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables having finite expectations. Prove that with probability one $\frac{X_n}{n} \to 0$ (i.e., $\frac{X_n}{n} \to 0$ almost surely).

b. Let P be the transition probability matrix of an ergodic (irreducible) Markov chain, and suppose $P^2 = P$. Prove that the chain is aperiodic and describe its stationary distribution. Is this chain reversible?

3. Linear and Integer Programming

Consider the cardinality constrained knapsack problem

$$\max \quad \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_j x_j \leq b \tag{1}$$

$$\text{at most } K \text{ variables can be positive} \tag{2}$$

$$x_j \leq 1, \qquad j \in \{1, \ldots, n\} \tag{3}$$

$$x_j \geq 0, \qquad j \in \{1, \ldots, n\}, \tag{4}$$

where $b \geq a_1 \geq \cdots \geq a_n > 0$.
Let $S = \{x \in \Re^n : x \text{ satisfies (1)-(4)}\}$, $PS = \text{conv}(S)$, and $LPS = \{x \in \Re^n : x \text{ satisfies (1), (3) and (4)}\}$.

1. Prove that PS is a polytope.
2. Prove that
$$\sum_{j=1}^{n} x_j \leq K \tag{5}$$
is a valid inequality for PS.
3. Prove that any vertex of LPS has at most one fractional component.
4. Prove that (5) cuts off every vertex of LPS which does not satisfy (2).
5. Let $C \subset \{1, \ldots, n\}$ be such that $|C| = K$ and $\sum_{j \in C} a_j > b$. Let $i \in C$ be such that $a_i \leq a_j \ \forall j \in C$. Let also $r \in \{1, \ldots, n\} - C$ be such that $\sum_{j \in C - \{i\}} a_j + a_r < b$. Prove that the inequality
$$\sum_{j \in C} a_j x_j + \left(b - \sum_{j \in C - \{i\}} a_j \right) x_r \leq b \tag{6}$$
is valid for PS.
6. Consider an instance with $n = 4, a_1 = 5, a_2 = 4, a_3 = 3, a_4 = 1, b = 7$, and $K = 2$. The point $\hat{x}_1 = \frac{1}{2}, \hat{x}_2 = 1, \hat{x}_3 = 0, \hat{x}_4 = \frac{1}{2}$ is a vertex of $LPS \cap \{x : x \text{ satisfies (5)}\}$. Derive an inequality (6) that cuts off $\hat{x}$.

4. Topology and Linear Algebra

Let Y be a subspace of a normed space X and let $x \in X$. Prove that $\operatorname{dist}(x, Y) \geq 1$ if and only if there exists a linear functional f on X of norm one such that Y is a subspace of the kernel of f and $f(x) = 1$. [$\operatorname{dist}(x, Y)$ is the infimum of $\|x - y\|$ over all $y \in Y$.]

5. Algebra

Suppose that F is an algebraically closed field. Show that F must be *infinite.*

Remark: A field must contain an additive inverse 0 and a multiplicative inverse 1 such that $1 \neq 0$. So the set $\{0\}$ is *not* a field.

6. Complexity

Show that, if the satisfiability problem can be decided by a BPP machine then $NP = RP$.

7. Combinatorial Methods

Prove that for sufficiently large n fewer than one half of all permutations of $\{1, 2, \ldots, n\}$ have a cycle of length 2.

8. Convexity

Let K denote a compact convex set in the plane $\mathbb{R}^2$. Recall that the *support function* $h_K : \mathbb{R}^2 \to \mathbb{R}$ of K is given by

$$h_K(x) = \max\{x \cdot y \mid y \in K\}.$$

If x is a non-zero vector in $\mathbb{R}^2$, let $x^\perp$ denote the line through the origin and orthogonal to the vector x. If K is a compact convex set, denote by $K|x^\perp$ the orthogonal projection of K onto the line $x^\perp$. Evidently $K|x^\perp$ is a closed interval in $x^\perp$. (Don't prove this.)

Define the *projection function* $f_K : \mathbb{R}^2 \to \mathbb{R}$ of K by

$$f_K(x) = |x| \cdot \text{length}(K|x^\perp),$$

where we define $f_K(0) = 0$. If x is a unit vector then $f_K(x)$ is equal to the *width* of K in the direction of $x^\perp$.

Problem: Show that f_K is the support function of another compact convex set L in $\mathbb{R}^2$. This set L is called the *projection body* of K, and is often denoted ΠK.

9. Design and Analysis of Algorithms

a) Let $G = (V, E)$ be an undirected connected graph with integral weights in the range $[1, 5]$ on its edges, and let $s \in V$ be a special vertex. Give an $O(m)$ algorithm for finding shortest paths for s to all the remaining vertices in the graph, where $|E| = m$.

b) Give an $O(n + m)$ time, i.e. linear time, algorithm to determine if a 2SAT formula is satisfiable; the formula has n variables and m clauses. By a 2SAT formula, we mean a Boolean formula in conjunctive normal form, with each clause having one or two literals.

c) Suppose you are given a subroutine for the following decision problem:

PARTITION

Instance: n positive integers, $a_1, \cdots a_n$.

Question: Is there a partition of these integers into two sets, A and B, that add up to the same amount?

Show how, using this subroutine, you can solve in polynomial time the search version of PARTITION, i.e., actually finding sets A and B that form a partition of the n integers, and add up to the same amount.