

1. Theory of Linear Inequalities

Let \mathbf{Z}_+ denote the positive integers. Suppose $a, b \in \mathbf{Z}_+^n$ are vectors and $\beta \in \mathbf{Z}_+$ is a scalar. Consider a matrix M where the columns of M are exactly the integer solutions to the knapsack problem ($a^T x \leq \beta, x \geq 0$). Let $\mathbf{1}$ denote the vector of all 1's. Show that the linear programming problem

$$\min \mathbf{1}^T y \quad (1)$$

$$\text{subject to} \quad (2)$$

$$My = b, \quad (3)$$

$$y \geq 0 \quad (4)$$

has an optimal solution if and only if $a_i \leq \beta$ for $i = 1, \dots, n$.

2. Combinatorial Optimization

Consider the following Arc Covering Problem (ACP): Given a directed graph $D = (N, A)$ with node set N , arc set A , integer arc costs $c(a)$ for $a \in A$, and a subset of arcs $L \subseteq A$, find a minimum cost set of simple directed cycles covering L (a cycle cover of L). Note that the cost of a cycle is the sum of the costs of its arcs.

1. Give a polynomial time algorithm for solving the ACP. (4pts)
2. Assume that you are given a fractional optimal solution to the ACP. Show that this fractional solution can be transformed into an integral optimal solution in polynomial time. (4pts)
3. Now assume there are two cost functions, c_1 and c_2 , given as input to the ACP. Define the cost of a cycle K to be $\min(c_1(K), c_2(K))$. Briefly discuss if the method you provided in question 2 will transform a fractional optimal solution to the two cost function ACP to an integral optimal solution in polynomial time. (2pts)

3. Analysis of Algorithms

(a) Given a complete bipartite graph $G = (U, V, U \times V)$ with non-negative weights on the edges and an integer $k > 0$, design a polynomial time algorithm to find a minimum weight matching of cardinality k . Solve this problem by reducing it to a single call of a minimum weight perfect matching algorithm.

(b) We have a graph $G = (V, E)$ with n vertices. Now suppose we would like to find a collection of matchings such that every edge of the graph is a member of (at least) one of the matchings selected. The goal is to pick the minimum number of matchings for our collection. Give an $O(\log n)$ -approximation algorithm for this problem. (Hint: Consider the set cover problem).

4. Randomized Algorithms

Consider the following random variable: X is 0 or 1 with equal probability. Let M be an n by n array of independent copies of X . Thus, M takes on all 2^{n^2} boolean values equally likely. Construct a

deterministic algorithm $A(s)$ that given a boolean string s of length $2n - 1$ outputs an n by n boolean matrix. The first, second, and third order statistics of $A(s)$ should be the same as those of M .

Thus, the algorithm $A(s)$ “simulates” a completely random boolean matrix with many fewer random bits. The simulation is only accurate up to the given statistics.

4. Graph Algorithms

A graph G is k -*extendable*, where $k \geq 1$ is an integer, if every matching of size at most k is a subset of a perfect matching of G . Design a polynomial-time algorithm for the following problem: Given a bipartite graph G and an integer $k \geq 1$, determine whether G is k -extendable. Prove correctness of your algorithm, and prove that it runs in polynomial time.

Hint. You may want to consider the connectivity of the directed graph obtained from G by directing all edges from A to B and contracting a perfect matching, where (A, B) is a bipartition of G .

5. Graph Theory

Let G be a 2-connected graph with vertex-set $\{1, 2, \dots, n\}$ containing a triangle, and let $\tau_1, \tau_2, \dots, \tau_{n-1}$ be distinct tokens. Initially, the tokens are placed on distinct vertices of the graph, at most one token per vertex. Thus exactly one vertex of G has no token on it. A *move* consists of sliding a token from its current position along an edge of G to a vertex that currently has no token on it. Prove that there exists a sequence of moves that results in token τ_i being positioned at vertex i for all $i = 1, 2, \dots, n - 1$.

6. Probability

1. Let X be a discrete time Markov chain with state space $S = \{1, 2\}$, and transition matrix

$$M = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}.$$

Classify the states of the Markov chain. Suppose that $a \cdot b > 0$ and $a \cdot b \neq 1$. Find the n -step transition probabilities and show directly that they converge to unique stationary distribution as $n \rightarrow \infty$.

2. Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities λ and μ . Show that the arrivals of these flying insects form a Poisson process with intensity $\lambda + \mu$.

7. Algebra

Let p, q be distinct primes. If a group G of order pq acts on a set X having $pq - p - q$ elements, show that there is a fixed point for this action (i.e., an element $x \in X$ with $g \cdot x = x$ for all $g \in G$).