1. Graph Theory

Let $G$ be a simple graph with maximum degree $d$. Prove that $E(G)$ can be decomposed into pairwise disjoint (possibly empty) matchings $M_1, \ldots, M_{d+1}$ such that $-1 \leq |M_i| - |M_j| \leq 1$ for all $1 \leq i, j \leq d+1$.

2. Probability

Assume that a Markov text $X_0, X_1, X_2, X_3, \ldots, X_n$ with letters from the alphabet $S = \{a, b, c\}$ has its transition probabilities given in the following transition matrix:

$$
\begin{pmatrix}
    p_{a\to a} & p_{a\to b} & p_{a\to c} \\
p_{b\to a} & p_{b\to b} & p_{b\to c} \\
p_{c\to a} & p_{c\to b} & p_{c\to c}
\end{pmatrix} = \begin{pmatrix}
    0.5 & 0.2 & 0.3 \\
    0.1 & 0.8 & 0.1 \\
    0.3 & 0.2 & 0.5
\end{pmatrix}
$$

a) What are the stationary probabilities $\pi(a), \pi(b), \pi(c)$ equal to? (i.e. what are the long term frequencies of the a’s, b’s and c’s in the text?)

b) Given that we start in state $a$, what is the expected time until we visit state $b$ for the first time?

3. Analysis of Algorithms

Given an undirected graph $G = (V, E)$, let $C$ be a coloring of $G$ where each vertex is colored either red, green, or blue. We say that an edge is monochromatic in $C$ if the colors on its endpoints agree, and bichromatic if they disagree. An ideal coloring is any coloring that maximizes the number of bichromatic edges. Unfortunately finding an ideal coloring is NP-hard.

(a) Let $M(G)$ be the number of monochromatic edges in an ideal coloring. Show that it is NP-hard to approximate $M(G)$ to within a factor of $10^{100}$.

(b) Let $B(G)$ be the number of bichromatic edges in an ideal coloring. Give a randomized algorithm that outputs a coloring such that the expected number of bichromatic edges is at least $\frac{2}{3}B(G)$.

4. Combinatorial Optimization

Let $A$ be an integral $m \times n$ matrix and let $b$ and $c$ be integral $m$-dimensional vectors. Show that there exists an integral vector $x$ with $Ax \in \{b, c\}$ if and only if there does not exist a vector $y$ such that $y^T A$ is integral, $y^T b$ is not an integer, and $y^T c$ is not an integer.

5. Theory of Linear Inequalities

Let $P = \{x : Ax \leq b, \ 0 \leq x \leq 1\}$ be a rational polytope of dimension $d$, where $A$ is an $m \times n$ matrix, $b$ is an $m$-dimensional vector, and $0$ and $1$ represent $n$-dimensional vectors with all components 0 and 1, respectively. Suppose $P$ does not contain any integer vectors. Show that the Chvátal rank of $P$ is no greater than $d$. 
6. Algebra

Let $G$ be a finite group acting on a set $X$, and let $H$ be a normal subgroup of $G$. If $x \in X$, show that the $G$-orbit of $x$ is a union of at most $|G:H|$ $H$-orbits of $X$, each having the same cardinality.

7. Randomized Algorithms

Consider the following scheme for shuffling a deck of $n$ cards labelled $c_1, c_2, \ldots, c_n$. For $i = 1, \ldots, n$, let $X_t(i)$ denote the card in the $i$-th position at time $t$. Let $X_0$ be an arbitrary ordering of the cards. For $t \geq 1$, given $X_{t-1}$ define $X_t$ as follows:

- Choose position $i$ uniformly at random from $\{1, \ldots, n\}$ and card $c_j$ uniformly at random from $\{c_1, \ldots, c_n\}$.
- Swap the card in position $i$ with card $c_j$. I.e., let $X_{t+1}(i) = c_j$ and let $X_{t+1}(k) = X_t(i)$ where $k = X_t^{-1}(c_j)$ is the position of card $c_j$ in $X_t$.
- For $\ell \notin \{i, k\}$, let $X_{t+1}(\ell) = X_t(\ell)$.

Show a coupling argument to upper bound the mixing time of this Markov chain, within a constant factor of optimal is fine. Recall, the mixing time is defined to be the number of steps (from the worst initial state) to get within variation distance $\leq 1/4$ of the uniform distribution.

7. Approximation Algorithms

Consider the following modification to the metric uncapacitated facility location problem. Define the cost of connecting city $j$ to facility $i$ to be $c_{ij}^2$. The $c_{ij}$’s still satisfy the triangle inequality (but the new connection costs, of $c_{ij}^2$, do not). Show that factor 3 primal-dual algorithm (given below), which uses the usual LP-relaxation and dual for the facility location problem, achieves an approximation guarantee of factor 9 for this case.

**Phase 1**

Raise the dual variable $\alpha_j$ for each unconnected city $j$ uniformly at unit rate, i.e., $\alpha_j$ will grow by 1 in unit time. When $\alpha_j = c_{ij}$ for some edge $(i, j)$, the algorithm will declare this edge to be tight. Henceforth, dual variable $\beta_{ij}$ will be raised uniformly, and it goes towards paying for facility $i$. Each edge $(i, j)$ such that $\beta_{ij} > 0$ is declared special. Facility $i$ is said to be paid for if $\sum_j \beta_{ij} = f_i$. If so, the algorithm declares this facility temporarily open. Furthermore, all unconnected cities having tight edges to this facility are declared connected and facility $i$ is declared the connecting witness for each of these cities. In the future, as soon as an unconnected city $j$ gets a tight edge to $i$, $j$ will also be declared connected and $i$ will be declared the connecting witness for $j$. When all cities are connected, the first phase terminates.

**Phase 2**

Let $F_t$ denote the set of temporarily open facilities and $T$ denote the subgraph of $G$ consisting of all special edges. Let $T^2$ denote the graph that has edge $(u,v)$ iff there is a path of length at most 2 between $u$ and $v$ in $T$, and let $H$ be the subgraph of $T^2$ induced on $F_t$. Find any maximal independent set in $H$, say $I$.

All facilities in the set $I$ are declared open. For city $j$, define $F_j = \{i \in F_t \mid (i, j) \text{ is special}\}$. Since $I$ is an independent set, at most one of the facilities in $F_j$ is opened. If there is a facility $i \in F_j$ that is
opened, then set $\phi(j) = i$. Otherwise, consider tight edge $(i', j)$ such that $i'$ was the connecting witness for $j$. If $i' \in I$, again set $\phi(j) = i'$. In the remaining case that $i' \notin I$, let $i$ be any neighbor of $i'$ in graph $H$ such that $i \in I$. Set $\phi(j) = i$. 