1. Computability, Complexity and Algorithms

Let $G = (V, E)$ be an undirected graph. Consider the following algorithm to find a large matching in $G$:

1. Start with $M = \emptyset$, the empty matching.
2. Add edges of $G$ greedily to $M$ as long as they maintain a matching.
3. If there is any edge $(u, v) \in M$ such that removing $(u, v)$ from $M$ allows you to add 2 new edges, then apply this change, increasing the size of $M$ by one. Repeat this step as long as such a change is possible (augmentations of length 3).

• Show that the resulting matching $M$ has at least $2/3$ as many edges as a maximum matching of $G$.

• Consider the extension where the algorithm augments on paths of length up to $2k + 1$. Show that the matching obtained has size at least $(k + 1)/(k + 2)$ times the size of the maximum cardinality matching.

• Suppose $G$ has nonnegative weights on its edges. Show that any greedy maximal matching — choose edges in order of weight while maintaining a matching — gives a matching of at least half the weight of a maximum weight matching.

2. Analysis of Algorithms

Given an edge weighted complete bipartite graph $G = (V, E)$ and a perfect matching $M$ in $G$, define $f(M)$ to be the weight of the heaviest edge in $M$. Define a bottleneck perfect matching in $G$ to be a perfect matching $N$ that minimizes $f(N)$.

First consider the algorithm that simply finds a minimum weight perfect matching in $G$. Give an example to show that the matching found by this algorithm may not be a bottleneck perfect matching. What is the approximation ratio achieved by this algorithm?

Give a polynomial time algorithm for finding a bottleneck perfect matching. Make sure your algorithm is as efficient as possible. What is its running time?

3. Theory of Linear Inequalities

Let $P \subseteq \mathbb{R}^n$ be a non-empty polytope. Let $\text{vert}(P)$ be the set of vertices of $P$. Let $X \subseteq \text{vert}(P)$. Define $P(X) := \text{conv}(\text{vert}(P) \setminus X)$. The graph of the polytope $P$ is a graph $G_P$ with nodes corresponding to $\text{vert}(P)$ such that two nodes are adjacent in $G_P$ if and only if the corresponding vertices are adjacent in $P$ (i.e. the two vertices are contained in a one-dimensional face of $P$).
Let $X \subseteq \text{vert}(P)$ and let $(X_1, \ldots, X_m)$ be a partition of $X$ such that $X_i$ and $X_j$ are independent in $G_P$, i.e., there is no edge connecting $X_i$ to $X_j$ for all $1 \leq i < j \leq m$. Then show that
\[ P(X) = \bigcap_{i=1}^{m} P(X_i). \]

4. Combinatorial Optimization

Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. Let $c(e)$ for $e \in E$ be the capacity of an edge. Furthermore, let $R = \{((s_1, t_1), d_1), ((s_2, t_2), d_2)\}$ be a set of two commodities, i.e., a quantity $d_1$ has to be send from source $s_1$ to sink $t_1$ and a quantity $d_2$ has to be send from source $s_2$ to sink $t_2$. Let $\delta_E(W)$ be the set of edges with exactly one endpoint in $W$ and let $\delta_R(W)$ be the set of commodities with either its source or its sink in $W$ but not both.

**Cut condition:** For each $W \subseteq V$, the capacity of $\delta_E(W)$ is not less than the demand of $\delta_R(W)$.

**Euler condition:**
\[
\sum_{e \in \delta(v)} c(e) \equiv 0 \pmod{2} \text{ if } v \neq s_1, t_1, s_2, t_2
\]
\[
d_1 \pmod{2} \text{ if } v = s_1, t_1
\]
\[
d_2 \pmod{2} \text{ if } v = s_2, t_2
\]

We have the following theorem:

**Theorem 1** If all capacities and demands are integer and both the cut condition and the Euler condition are satisfied, then the undirected 2-commodity flow problem has an integer solution.

**Question 1.** Prove the following lemma

**Lemma 1** Every cut in an Eulerian graph (with edge capacities equal to one) has even cardinality.

**Question 2.** Use Theorem 1 and Lemma 1 to show the following. Let $G = (V, E)$ be an Eulerian graph and let $s_1, t_1, s_2, t_2$ be distinct vertices. Then the maximum number $k$ of pairwise edge-disjoint paths $P_1, \ldots, P_k$, where each path $P_j$ connects either $s_1$ and $t_1$ or $s_2$ and $t_2$, is equal to the minimum cardinality of a cut both separating $s_1$ and $t_1$ and separating $s_2$ and $t_2$. 
5. Graph Theory

Let $k \geq 1$ be an integer and let $G$ be a $k$-connected $k$-regular graph on an even number of vertices. Prove that $G$ has a perfect matching.

6. Probabilistic methods

Let $X_1, \ldots, X_n$ be independent random variables with $X_i \in \{0, 1\}$ and $\text{Prob}[X_i = 1] = p$, for $i = 1, \ldots, n$, where $0 < p < 1$. Set $X := \sum_{i=1}^{n} X_i$. Prove that for any $t \in [0, 1 - p]$, we have

$$\text{Prob}[X \geq (p + t)n] \leq e^{-nh(p, t)},$$

where $h(p, t) = (p + t) \ln \frac{p + t}{p} + (1 - p - t) \ln \frac{1 - p - t}{1 - p}$, and is also referred to as a “relative entropy function”.

7. Algebra

(a) Suppose $K \subset H \subset G$ are groups under the same operation and that $K$ is normal in $H$ and $H$ is normal in $G$. Does $K$ have to be normal in $G$?

(b) Let $G$ be a group and $H$ be a subgroup of $G$ with index $n$. Prove that there is a normal subgroup $K$ of $G$ such that $K \subset H$ and $[G : K] \leq n!$.

7. Linear Algebra

Let $T \in \text{Hom}(V, V)$, where $V$ is an $n$-dimensional vector space over a field $\mathbb{F}$. (In other words, $T$ is a linear transformation from $V$ to $V$.)

(i) Show that if $T^m = 0$, but $T^{m-1} \neq 0$, then there is a vector $v \in V$ such that $\{v, Tv, \ldots, T^{m-1}v\}$ is a linear independent set.

(ii) Show that if $T^m = 0$, then $T^n = 0$.

(iii) Show that if $\ker(T) \cap \text{Im}(T) = \{0\}$, then $\ker(T^2) = \ker(T)$. By giving an example, show that the conclusion is false if the assumption $\ker(T) \cap \text{Im}(T) = \{0\}$ does not hold.