1. Analysis of Algorithms

Describe an algorithm for deciding if an $n$-vertex graph $G$ contains a clique of size $6$. Explain how to modify the algorithm so it would also find such a clique in $G$ (if one exists). The running time of both algorithms should be $O(n^5)$.

**Hint.** You may wish to consider a graph with vertex-set $O$

**Solution:** Given $G$ let $m = |E|$ and define an $m$-vertex graph $T$ as follows. Each vertex of $T$ represents an edge of $G$. We connect two vertices $(u, u'), (v, v')$ of $T$ if and only if $u, u', v, v'$ form a clique of size $4$ in $G$. Then it is easy to see that $G$ contains a clique of size $6$ if and only if $T$ contains a triangle. Now we can use fast matrix multiplication to decide in time $O(m^\omega) = O(n^{2\omega}) \ll O(n^5)$ if $T$ contains a triangle. In order to actually find such a triangle, we can use the algorithm we saw in class that finds witnesses for Boolean matrix multiplication in time $O(n^\omega)$.

2. Approximation Algorithms

Let $G = (V, E)$ be a complete graph with distances on its edges; the distance between two vertices $u$ and $v$ is given by $d(u, v)$ and the distances satisfy the triangle inequality. The $k$-partition problem is to partition $V$ into $k$ subsets, $C_1, C_2, \ldots, C_k$, so that the maximum distance between any pair of vertices in the same subset is minimized. Formally, define the diameter of a subset $C_r \subset V$ as

$$Diam(C_r) = \max\left\{ d(v_i, v_j) : v_i, v_j \in C_r \right\}$$

Then we wish to find a partition that minimizes $\max_{r \in \{1, 2, \ldots, k\}} Diam(C_r)$.

Now consider the following process. Start with an arbitrary vertex. Call it $v_1$. Then at the $i^{th}$ step, $i \geq 2$, let

$$\delta_i = \max_u \min_{j \in \{1, 2, \ldots, i-1\}} d(u, v_j)$$

and define $v_i$ to be the vertex $u$ that achieves the maximum. That is, $v_i$ is the vertex $u$ that maximizes the minimum distance of $u$ to one of the vertices in $\{v_1, v_2, \ldots, v_{i-1}\}$.

1. Show that $\delta_{k+1}$ is a lower bound on the value of the optimal solution of the $k$-partition problem.

2. Give an efficient 2-approximation algorithm for the $k$-partition problem.

**Solution.** Note that $\delta_i$ is a nondecreasing sequence. Let $v_1, \ldots, v_{k+1}$ be the first $k + 1$ points in the sequence found by the process described. Consider the partition on them induced by the optimal $k$-partition. At least two of the vertices, say $v_i$ and $v_j$, with $i < j$, must be in the same part of the optimal partition. This implies that the diameter of the part they lie in must be at least $\delta_j$. Therefore, the optimal partition has at least one part of diameter at least $\delta_{k+1}$, i.e., $OPT \geq \delta_{k+1}$.

For the second part, find the first $k$ vertices according to the process. Call these the anchors of a $k$-partition. For every vertex $u \notin \{v_1, \ldots, v_k\}$, assign it part $i$ if $v_i$ is the closest to $u$ among the anchors (break ties arbitrarily). Then for each part $i$, for any vertex $u$ in the part, $d(u, v_i) \leq \delta_{k+1}$. Therefore, by the triangle inequality, for any two vertices $u, v$ in the same part $i$,

$$d(u, v) \leq d(u, v_i) + d(v_i, u) \leq 2\delta_{k+1} \leq 2OPT.$$
3. Theory of Linear Inequalities

Let \( d, f \in \mathbb{R}^n \) be integer vectors with all components positive and let \( t \) be a positive integer. Suppose \( d_i \leq t \) for all \( i = 1, \ldots, n \), where \( d = (d_1, \ldots, d_n)^T \). Let \( A \) be a matrix such that columns of \( A \) are the non-negative integer solutions to the inequality \( d^T x \leq t \). The integer cutting-stock problem is

\[
\min(e^T y : Ay = f, y \geq 0, y \text{ integer})
\]

(1)

where \( e \) is the vector of all 1’s. Show that (1) has an optimal solution with at most \( 2^n \) positive components.

Solution. A solution is available upon request.

4. Combinatorial Optimization

Let \( G = (V, E) \) be a complete graph having an even number of vertices and let \( c = (c_e : e \in E) \) be edge weights such that \( c \geq 0 \) and \( c \) satisfies the triangle inequality. For \( X \subseteq V \) let \( \delta(X) \) denote the set of edges with one end in \( X \) and the other end in \( V - X \). Let \( \mathcal{C} \) denote the set of all sets \( D \) of the form \( D = \delta(X) \) such that \( X \subseteq V, |X| \geq 3, |V(G) - X| \geq 3 \) and \( |X| \) is odd. The dual LP for Edmonds’ perfect-matching system is

\[
\begin{aligned}
\text{Maximize } & \sum (y_v : v \in V) + \sum (Y_D : D \in \mathcal{C}) \\
\text{subject to } & y_v + y_w + \sum (Y_D : e \in D \in \mathcal{C}) \leq c_e, \text{ for all } e = vw \in E \\
& Y_D \geq 0, \text{ for all } D \in \mathcal{C}.
\end{aligned}
\]

Show that there exists an optimal dual solution such that \( y_v \geq 0 \) for all \( v \in V \).

Solution. A solution is available upon request.

5. Graph Theory

Let \( k \geq 2 \) be an integer. Prove that in a \( k \)-connected graph, for every set of \( k \) vertices there is a cycle that includes all of them.

Solution: For \( k = 2 \) this follows directly from Menger’s theorem. For \( k > 2 \) there is, by induction, a cycle \( C \) containing \( k - 1 \) of the given vertices, and we may assume that the last vertex, say \( v \), is not on \( C \). The \( k - 1 \) given vertices on \( C \) divide \( C \) into \( k - 1 \) edge-disjoint paths. Let us call those paths segments. If \( |V(C)| = k - 1 \) (that is, \( V(C) \) consists entirely of the given vertices), then let \( l := k - 1 \); otherwise let \( l := k \). By Menger’s theorem there exist \( l \) paths from \( v \) to \( V(C) \), vertex-disjoint, except for \( v \). It follows that some two of those paths, say \( P \) and \( Q \), have ends in the same segment, and hence \( C \cup P \cup Q \) contains a cycle that includes all the given vertices.

6. Probabilistic methods

Let \( G = (V, E) \) be a graph with \( n \) vertices and \( m \) edges. Let \( t \geq 1 \) be arbitrary.
(i) Form a (random) subset $T$ of $V(G)$ by picking a (uniformly) random vertex of the graph $t$ times, with repetition. (Thus $|T| \leq t$.) Let $N(T)$ denote its common neighborhood – the set of vertices adjacent to every vertex of $T$. Let $X = |N(T)|$.

Show that $E[X] \geq \frac{(2m)^t}{n^{2t-1}}$.

(ii) Suppose that
$$\frac{(2m)^t}{n^{2t-1}} - \binom{n}{s} \left( \frac{k}{n} \right)^t \geq u.$$ Then prove that there exists a subset $U \subset V(G)$ of at least $u$ vertices, such that every set of $s$ vertices in $U$ has at least $k$ common neighbors.

Solution: (i) Note that the probability that a vertex $v$ is in $N(T)$ is just the probability that $T$ is a subset of its neighborhood. Hence, by the convexity of $x^t$ (for $t \geq 1$),
$$E(X) = \sum_{v \in V} \left( \frac{|N(v)|}{n} \right)^t \geq n \left( \frac{1}{n} \sum_{v \in V} \frac{|N(v)|}{n} \right)^t = \frac{(2m)^t}{n^{2t-1}}.$$ (ii) (Use the deletion method.) Let $A := N(T)$. Let $Y$ denote the number of $s$-sets in $A$ with at most $k$ common neighbors. Suppose the pair $\{u, v\}$ has at most $k$ common neighbors; then the probability that a $\{u, v\} \subset A$ is at most $\left( \frac{k}{n} \right)^t$, since each element of $T$ must lie in the common neighborhood of $u$ and $v$; the same argument holds for subsets of $s$ vertices, rather than pairs. And so
$$E(Y) \leq \binom{n}{s} \left( \frac{k}{n} \right)^t.$$ By linearity of expectation,
$$E[X - Y] \geq \frac{(2m)^t}{n^{2t-1}} - \binom{n}{s} \left( \frac{k}{n} \right)^t \geq u,$$ and thus there must exist a choice of $T$ such that $X - Y \geq u$. (As usual), simply remove one element from each $s$-set in $A$ with at most $k$ neighbors, to obtain $U$ as required.

7. Algebra

Prove that any finite subgroup of the multiplicative group of a field is cyclic.

Solution: Let $F$ be a field and $G$ be a finite subgroup of the group $F^\times = F \setminus \{0\}$ under multiplication. Since $G$ is finite and abelian, by the Structure Theorem for Abelian Groups, $G$ is a direct product of finitely many cyclic groups, i.e. $G \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ for some integers $n_1, n_2, \ldots, n_k \geq 2$. It suffices to show that $\gcd(n_i, n_j) = 1$ if $i \neq j$. For $i \neq j$, suppose there is a prime $p$ dividing both $n_i$ and $n_j$. Then it follows from Sylow Theorem that $C_{n_i}$ and $C_{n_j}$ both contain elements of order $p$. Since $p$ is prime, if $a$ has order $p$, then so does $a^2, \ldots, a^{p-1}$. Hence both $C_{n_i}$ and $C_{n_j}$ contain at least $p - 1$ elements of order $p$. However, in a field $F$, the polynomial $x^p - 1$ has at most $p - 1$ roots other than 1, so $C_{n_i}$ and $C_{n_j}$ have a non-empty intersection, which cannot happen in a direct product.