1. Computability, Complexity and Algorithms

(a) Let $G(V, E)$ be an undirected unweighted graph. Let $C \subseteq V$ be a vertex cover of $G$. Argue that $V \setminus C$ is an independent set of $G$.

(b) Minimum cardinality vertex cover and maximum cardinality independent set are well known NP-complete problems. Suppose that you have a polynomial time approximation algorithm that, on input an undirected unweighted graph $G(V, E)$, outputs a vertex cover $C$ whose cardinality is at most $2OPT$. Is the cardinality of the independent set $V \setminus C$ a constant factor approximation algorithm for maximum independent set? If yes give a proof, if no give a counter example.

(c) Give a polynomial time algorithm that, on input an undirected unweighted bipartite graph $G$, outputs a minimum cardinality vertex cover of $G$.

2. Analysis of Algorithms

1. The exact matching problem is the following: Given a bipartite graph $G = (U, V, E)$ and an integer $k \leq n$, with $|U| = |V| = n$ and with a subset $E' \subseteq E$ of the edges colored red, an exact matching is a perfect matching with exactly $k$ red edges. Give randomized polynomial-time algorithms for:

   (a) Testing if $G$ has an exact matching.

   (b) If so find one.

2. Next consider an extension of this problem where two disjoint subsets $E_1$ and $E_2$ of edges are colored red and blue, respectively, and two integers $k_1, k_2$ are specified with $k_1 + k_2 \leq n$. Now we seek a perfect matching with $k_1$ red edges and $k_2$ blue edges. Repeat the two previous questions for this extended notion.

3. Theory of Linear Inequalities

For a system $Ax \leq b$ of $m$ rational linear inequalities and a set $S \subseteq \{1, \ldots, m\}$ let

$$A_S x = b_S, \quad A_{\bar{S}} x \leq b_{\bar{S}}$$

(1)

denote the system obtained by setting each inequality in $S$ to equality while keeping each inequality in $\bar{S} = \{1, \ldots, m\} \setminus S$ as an inequality.

Suppose the system (1) has no solution for some specified set $S$. Then, by Farkas’s Lemma, there exists a vector $(y_S, y_{\bar{S}})$ such that

$$y_S^T b_S + y_{\bar{S}}^T b_{\bar{S}} < 0, \quad y_S^T A_S + y_{\bar{S}}^T A_{\bar{S}} = 0, \quad y_{\bar{S}} \geq 0.$$  (2)
Due to the equality constraints, the vector $y_S$ might have negative components. Notice, however, that we may scale $(y_S, y_S)$ so that it satisfies

$$y_S^Tb_S + y_S^Tb_S < 0, \ y_S^TA_S + y_S^TA_S = 0, \ y_S \geq -1, \ y_S \geq 0.$$ 

(3)

Now (2) necessarily has an integral solution (since it has a rational solution), but (3) may not be solvable in integers. We say that the infeasibility of (1) can be proven integrally if (3) does in fact have an integral solution.

Prove the following theorem.

**Theorem 1** Let $A$ be an integral matrix and let $b$ be a rational vector such that $Ax \leq b$ has at least one solution. Then $Ax \leq b$ is totally dual integral if and only if

(i) the rows of $A$ form a Hilbert basis

and

(ii) for each subset $S$ of inequalities from $Ax \leq b$, if (1) is infeasible, then this can be proven integrally.

**4. Combinatorial Optimization**

Recall that a graph $G$ is factor critical if for all $v \in V(G)$, $G - v$ has a perfect matching. A near perfect matching is a matching covering all but one vertex of the graph. It is known that every 2-connected factor critical graph $G$ contains pairwise edge-disjoint subgraphs $G_0, H_1, \ldots, H_k$ satisfying the following. For $j = 1, \ldots, k$, let $G_j = G_0 \cup \bigcup_{i=1}^{j} H_i$.

a. $G_0$ is an odd cycle and $G = G_k$,

b. $H_i$ is an odd length path with both ends in $G_{i-1}$ and no internal vertex in $V(G_{i-1})$.

Specifically, the endpoints of $H_i$ are distinct.

You may use this assertion without proof. Show that every 2-connected factor critical graph $G$ contains at least $|E(G)|$ distinct near perfect matchings.

**5. Graph Theory**

Prove that for every integer $k \geq 1$ there exists an integer $N$ such that if the subsets of \{1, 2, \ldots, N\} are colored using $k$ colors, then there exist disjoint non-empty sets $X, Y \subseteq \{1, 2, \ldots, N\}$ such that $X, Y$ and $X \cup Y$ receive the same color.

**Hint.** You may want to consider intervals.
6. Probabilistic methods

A proper list-coloring of a graph $G = (V, E)$ from lists $\{L_v \subseteq \mathbb{N} \mid v \in V\}$ is a function $c : V \to \mathbb{N}$ such that $c(v) \in L_v$ for all $v \in V$ and $c(u) \neq c(v)$ for all $\{u, v\} \in E$.

Let $r$ be a natural number. Prove that if for all $v \in V$ we have $|L_v| = 10r$ and for all $j \in L_v$ there are at most $r$ neighbors $u \in V$ of $v$ such that $j \in L_u$, then $G$ admits a proper list-coloring from these lists.

7. Algebra

Two polynomials $f, g \in R[t]$ over a commutative ring $R$ with identity are called relatively prime over $R$ if $f$ and $g$ generate the unit ideal in $R[t]$. Let $f, g \in \mathbb{Z}[t]$ be non-constant monic polynomials such that $f$ and $g$ are relatively prime over $\mathbb{Q}$ and the residues of $f$ and $g$ modulo $p$ are relatively prime over $\mathbb{Z}/p\mathbb{Z}$ for all prime numbers $p$. Prove that $f$ and $g$ are relatively prime over $\mathbb{Z}$. 