COMBINATORIAL PROBLEMS FOR GRAPHS AND PARTIALLY ORDERED SETS

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in
Algorithms, Combinatorics, and Optimization

School of Mathematics
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December 2015

COMBINATORIAL PROBLEMS FOR GRAPHS AND PARTIALLY ORDERED SETS

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Date Approved: 9 November 2015

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SUMMARY

This dissertation has three principal components. The first component is about the connections between the dimension of posets and the size of matchings in comparability and incomparability graphs. In 1951, Hiraguchi proved that for any finite poset P, the dimension of P is at most half of the number of points in P. We develop some new inequalities for the dimension of finite posets. These inequalities are then used to bound dimension in terms of the maximum size of matchings. We prove that if the dimension of P is d and d is at least 3, then there is a matching of size d in the comparability graph of P, and a matching of size d in the incomparability graph of P. The bounds in above theorems are best possible, and either result has Hiraguchi's theorem as an immediate corollary.

In the second component, we focus on an extremal graph theory problem whose solution relied on the construction of a special kind of posets. In 1959, Paul Erdős, in a landmark paper, proved the existence of graphs with arbitrarily large girth and arbitrarily large chromatic number using probabilistic method. In a 1991 paper of Kříž and Nešetřil, they introduced a new graph parameter eye(G). They show that there are graphs with large girth and large chromatic number among the class of graphs having eye parameter at most three. Answering a question of Kříž and Nešetřil, we were able to strengthen their results and show that there are graphs with large girth and large chromatic number among the class of graphs having eye parameter at most two.

The last component is about random poset—the poset version of the Erdős–Rényi random graph. In 1991, Erdős, Kierstead and Trotter (EKT) investigated random height 2 posets and obtained several upper and lower bounds on the dimension of the random posets. Motivated by some extremal problems involving conditions which force a poset to contain a large standard example, we were compelled to revisit this subject. Our sharpened analysis allows us to conclude that as p approaches 1, the expected value of dimension first increases and then decreases, a subtlety not identified in EKT. Along the way, we establish connections with classical topics in analysis as well as with latin rectangles. Also, using structural insights drawn from this research, we are able to make progress on the motivating extremal problem with an application of the asymmetric form of the Lovász Local Lemma.

CHAPTER I

INTRODUCTION

In this chapter, we introduce some basic notation and terminology for graphs and partially ordered sets.

1.1 Basic notation for graphs

A (simple) *graph* G is an ordered pair (V, E), where V is a set and E is a 2-elements subset of V. V is called the *vertex set* of G, and E is called the *edge set* of G. The elements of V are the *vertices* of G, the elements of E are the *edges* of G. A graph is *finite* if it contains finite number of vertices. In this dissertation, we focus only on finite graphs.

Given a graph G, a vertex v of G is *incident* with an edge e if $v \in e$. An edge $\{x, y\}$ in E is usually written as xy. Two vertices x, y of G are *adjacent* if xy is an edge of G. If all the vertices of G are pairwise adjacent, then G is called a *complete graph*.

We say $G' = (V', E')$ is a *subgraph* of $G = (V, E)$, denoted by $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. G' is an *induced subgraph* of G if $G' \subseteq G$ and G' contains all the edge $xy \in E$ with $x, y \in V'$.

Let $N_G(v)$ (or simply $N(v)$) denote the set of neighbors of a vertex v in G, i.e., $N_G(v) = \{u : uv \text{ is an edge in } G\}$. The *degree* of a vertex v, denoted by $\deg_G(v)$ (or simply $\deg(v)$), is equal to $|N(v)|$. A vertex of degree 0 is said to be *isolated.* $\delta(G)$ and $\Delta(G)$ are, respectively, the *minimum degree* and the *maximum degree* of vertices in G.

A *path* is a non-empty graph $P = (V, E)$ where V consists of distinct vertices $v_0, v_1, \ldots, v_n$ and $E = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}$. Such path is called a v_0–v_n path. The *length* of a path is the number of edges it contains. A *cycle* is a closed path where $v_0 = v_n$. A graph is *acyclic* if it does not contain any cycles. Similarly the *length* of a cycle is the number of edges it contains.

Given a graph G, the *distance* $d_G(x, y)$ (or simple $d(x, y)$) of two vertices x and y is the length of the shortest x–y paths in G. If there is no x–y path in G, then $d(x, y) = \infty$. A graph G is *connected* if every pair of vertices of G has finite distance.

An acyclic graph is called a *forest*. A connected forest is a *tree*. The degree 1 vertices of a tree are *leaves*.

For additional background material on graph theory, the readers are referred to Diestel's text [13].

1.2 Basic notation for partially ordered sets

A *partially ordered set*, or a *poset* P, is an ordered pair (X, R), where X is a set and R is a reflexive, antisymmetric and transitive binary relation defined on X. We call X the *ground set* and R is a *partial order* on X. Elements of X are called *points*. We write $x < y$ or $y > x$ when $(x, y) \in R$. The poset is *finite* if the ground set X is a finite set. In this dissertation, we focus only on finite posets.

Given a poset P and $x, y \in P$ with $x \neq y$, we say x and y are *incomparable* in P, denoted by $x \parallel y$, if $x \not> y$ and $y \not> x$ in P. Define $\mathrm{Inc}(P)$ to be the set of all incomparable pairs in P. Note that $\mathrm{Inc}(P)$ is a symmetric, irreflexive binary relation. We say x and y are *comparable* in P, denoted by $x \perp y$, if either $x > y$ or $y > x$ in P.

A point $x \in P$ is called a *minimal* element (*maximal* element, resp.) if there is no $y \in P$ such that $y < x$ ($x < y$, resp.) in P. We denote $\mathrm{Min}(P)$ ($\mathrm{Max}(P)$, resp.) be the set of all minimal elements (maximal elements, resp.) in P. We say $Q = (Y, S)$ is a *subposet* of $P = (X, R)$, if Y is a non-empty subset of X, and the restriction of R on Y is S.

A poset P is called a *chain*, if x and y are comparable for any two distinct points

$x, y \in P$. Dually, A poset P is called an *antichain*, if x and y are incomparable for any two distinct points $x, y \in P$. A chain C is a *maximum chain* in P if there are no chains in P contain more points than C. A chain C is a *maximal chain* in P if there are no other chains in P contain C. *Maximum* and *maximal antichains* are defined dually. The *height* of P is the number of points in a maximum chain in P, while the *width* of P is the number of points in a maximum antichain in P.

When $x \in P$, we write $D(x)$ for the set $\{y \in P : y < x \text{ in } P\}$, while $U(x)$ is the set $\{y \in P : y > x \text{ in } P\}$. When A is a maximal antichain in P, $D(A)$ consists of all $x \in P$ for which there is some $a \in A$ with $x < a$ in P. Dually, $U(A)$ consists of all $x \in P$ for which there is some $a \in A$ with $x > a$ in P. Evidently, $D(A) \cap U(A) = \emptyset$.

Given a poset P and two distinct points $x, y \in P$, we say x is *covered by* y (or y *covers* x) if $x < y$ in P, and there is no $z \in P$ such that $x < z < y$ in P. We can then associate with the poset P a *cover graph* G on the same ground set as P, so that xy is an edge in G if and only if x covers y in P or y covers x in P.

It is convenient to specify a finite poset by a suitably drawn diagram of the cover graph in the Euclidean plane. Whenever $x < y$ in P, we require the vertical coordinate of x be smaller than the vertical coordinate of y. Such diagrams are called *Hasse diagrams* (or *order diagrams*).

In Figure 1, we illustrate an order diagram of a poset P on 15 points with *height*$(P) = 5$ and $\text{width}(P) = 6$. Observe that:

1. a is covered by g in P;

2. $a < c$ in P;

3. $k \parallel c$ in P;

4. $k \perp m$ in P;

5. $\text{Max}(P) = \{c, d, e, g, k, l\}$ and $\text{Min}(P) = \{a, h, j, m\}$;

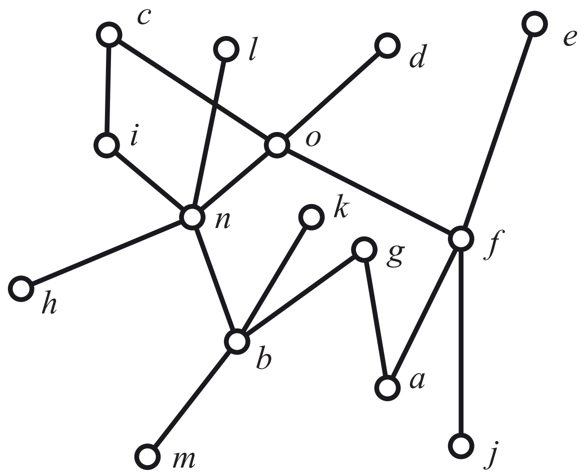

Figure 1: An order diagram of a poset P on 15 points

6. $U(a) = \{c, d, e, f, g, o\}$;

7. $D(g) = \{a, b, m\}$;

8. $\{e, f, j\}$ is a maximal chain;

9. $\{m, b, n, i, c\}$ is a maximum chain;

10. $\{n, k, g, f\}$ is a maximal antichain;

11. $\{c, l, d, e, k, g\}$ is a maximum antichain.

CHAPTER II

DIMENSION AND MATCHINGS IN COMPARABILITY AND INCOMPARABILITY GRAPHS

2.1 Introduction

In this chapter, we focus on combinatorial problems associated with order diagrams, comparability graphs and incomparability graphs.

When P is a poset, the *comparability graph* G_P of P is the graph whose vertex set is the ground set of P with xy an edge in G_P if and only if x and y are distinct comparable points in P. Dually, the *incomparability graph* H_P of P is just the complement of G_P, i.e. xy is an edge in H_P if and only if x and y are distinct incomparable points in P.

In Figure 2, G_P and H_P are, respectively, the comparability graph and the incomparability graph of poset P on the left hand side.

Dimension is one of the most important notions in poset theory. The *dimension* of a poset P, denoted $\dim(P)$, is the least positive integer d for which there is a family $\mathcal{R} = \{L_1, L_2, \ldots, L_d\}$ of linear extensions of P so that $x \leq y$ in P if and only if $x \leq y$

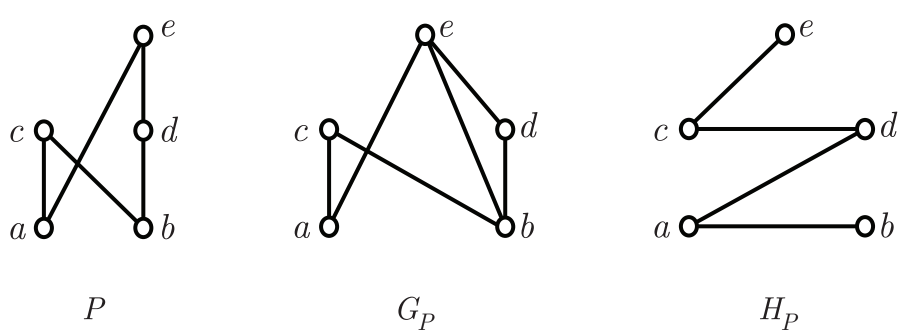

Figure 2: Comparability graph and incomparability graph

in L_i for each $i \in \{1, 2, \ldots, d\}$. Alternatively, the *dimension* of P is the least positive integer d for which there is a family $\mathcal{R} = \{L_1, L_2, \ldots, L_d\}$ of linear extensions of P so that $P = L_1 \cap L_2 \cap \cdots \cap L_d$.

Dimension is *monotone*, i.e., if Q is a subposet of P, then $\dim(Q) \leq \dim(P)$. The next result, due to Hiraguchi [24], is fundamental to our subject.

Theorem 2.1.1. *Let P be a poset with at least two points. If $x \in P$, then* $\dim(P) \leq 1 + \dim(P - \{x\})$.

We view the preceding theorem as asserting that dimension is "continuous", i.e., small changes in the poset can only make small changes in dimension.

Let P be a poset. $\dim(P) = 1$ if and only if P is a chain. Clearly a chain has dimension 1. On the other hand, an antichain of size two or larger has dimension 2. So P does not contain any antichains.

We use the notation P^* for the *dual* of P, i.e., P^* has the same ground set as P with $x > y$ in P^* if and only if $x < y$ in P. Similarly, when L is a linear extension of P, let L^* denote the *dual* of L with the property that $x > y$ in L if and only if $x < y$ in L^*. Note that L is a linear extension of P if and only if L^* is a linear extension of P^*. It then implies $\dim(P) = \dim(P^*)$.

The following result is due to Dushnik and Miller [16].

Theorem 2.1.2. *Let P be a poset. Then* $\dim(P) \leq 2$ *if and only if the incomparability graph of P is also a comparability graph.*

Note that if poset $Q = L_1 \cap L_2^*$ where $\mathcal{R} = \{L_1, L_2\}$ is a realizer of P, then the incomparability graph of P is the comparability graph of Q.

A poset P (with at least two points) is *irreducible* when $\dim(P - \{x\}) < \dim(P)$ for every x in P. An irreducible poset P is *d-irreducible* if $\dim(P) = d$. The only 1-irreducible poset is a single point. The only 2-irreducible poset is a 2-element antichain. A full listing of all 3-irreducible posets has been assembled by Kelly [27]

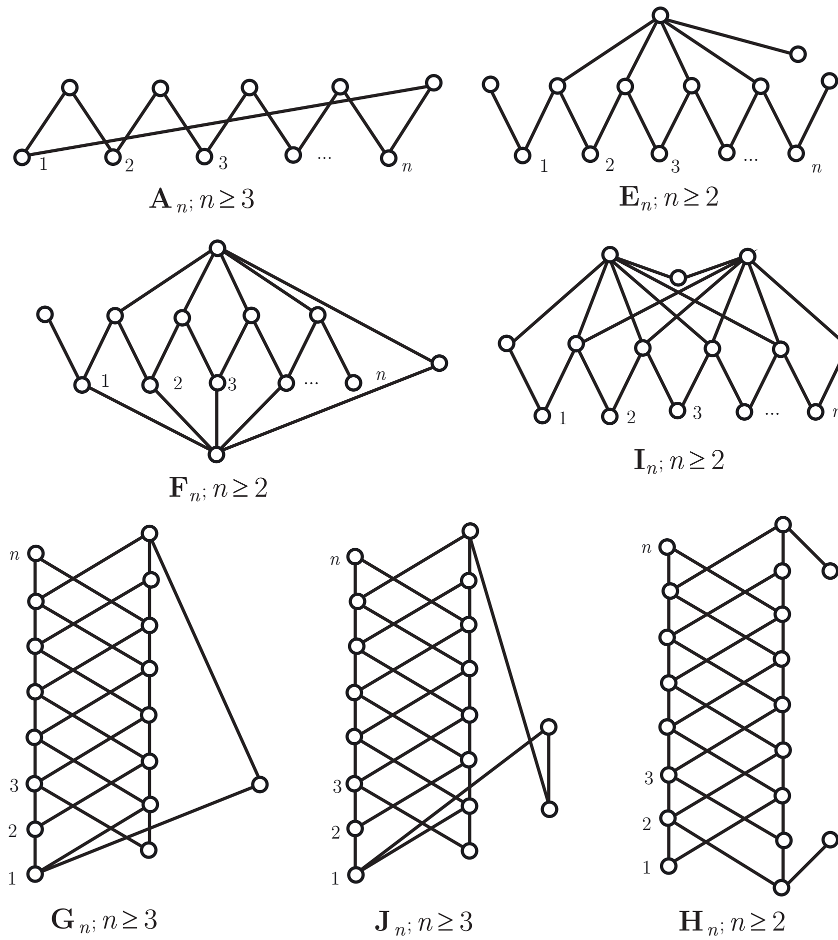

Figure 3: Families of 3-irreducible posets

and by Trotter and Moore [45]. These posets are illustrated in Figures 3 and 4. The posets shown in Figure 3 constitute seven infinite families, while the posets shown in Figure 4 are "miscellaneous" examples. In cases where a 3-irreducible poset is not self-dual, only one of the two instances is included in these figures.

When $n \geq 8$, inspection of the posets illustrated in Figures 3 and 4 shows that the number of 3-irreducible posets on n points is at most 7 (all posets in Figure 4 have at most 7 points). On the other hand, for each $d \geq 4$, Trotter and Ross [46] showed that there is a constant $c_d > 1$ and an integer n_d so that for $n \geq n_d$, the number of d-irreducible posets on n points is more than $c_d^{n^2}$. As a consequence, there are no complete listings for d-irreducible posets when $d \geq 4$.

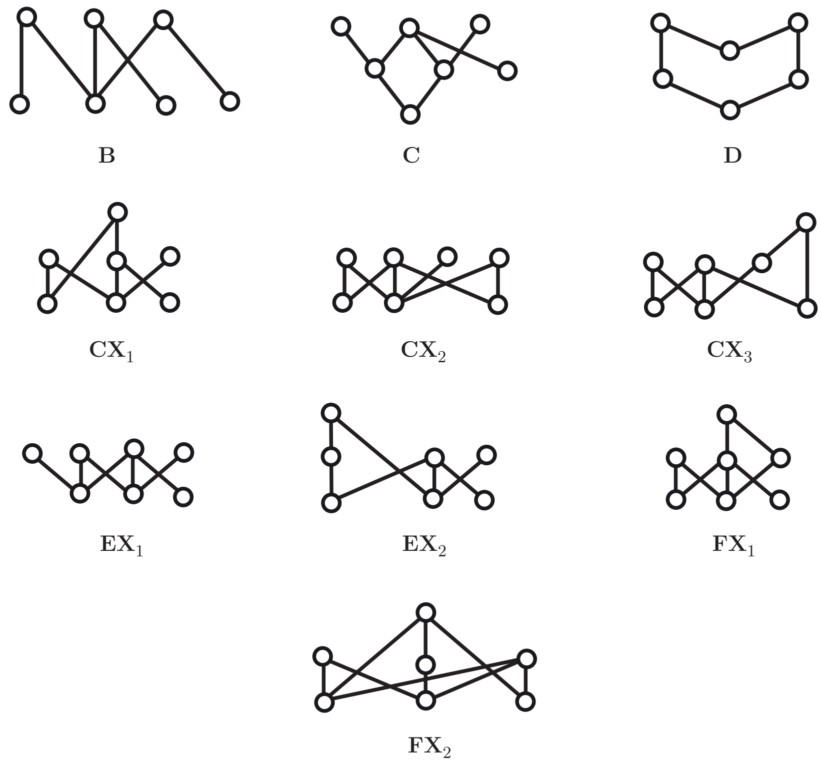

Figure 4: Miscellaneous examples of 3-Irreducible posets

The remainder of this chapter is organized as follows. In the next section, we provide some essential background material, and introduce Hiraguchi's inequality. In Section 2.3, we provide a brief sketch of related work which serves to motivate the research reported here. In section 2.4, we introduce our main theorems–one chain matching theorem and one antichain matching theorem. In Section 2.5, we prove three new inequalities for dimension. The first is elementary, but the second and third are more substantive, and they are the key ingredients of the proofs for our matching theorems. In Section 2.6, we discuss chain matchings and prove the chain matching theorem. At the close of this section, we explain why there is no analogue for matchings in the cover graph. In Section 2.7, we prove the antichain matching theorem.

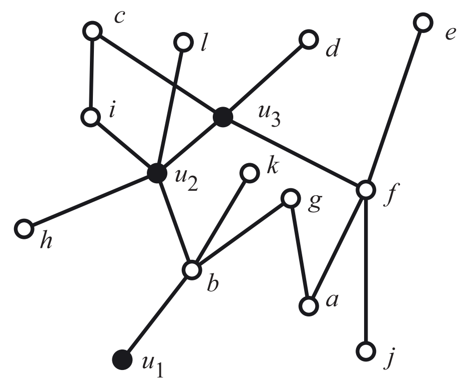

Figure 5: Defining linear extensions using blocks

2.2 Hiraguchi's Inequality

Before introducing the Hiraguchi's Inequality, we need the following notation.

When P is a poset, we use $|P|$ to denote the number of elements in the ground set of P. Subposets of P are identified just by specifying their ground sets. For example, if x and y are distinct elements of P, $P - \{x, y\}$ is just the subposet obtained when x and y are removed from P. When S is a subposet of P and L is a linear extension of P, $L(S)$ denotes the restriction of L to S.

Define *blocks* to be disjoint subsets of vertex set of P. Blocks will be used to define a linear extension of P. In Figure 5, we show a poset P. For this poset, define blocks $B_1 = \{a, e, f, h, j\}$, $B_2 = \{b, g, k\}$, $B_3 = \{i, l\}$ and $B_4 = \{c, d\}$. Then set $L_2 = [j < a < b < k < h < g]$ and $L_4 = [b < k < d < c]$. In this dissertation, we use notation such as

$$L = [B_1 < u_1 < L_2(B_2) < u_2 < B_3 < u_3 < L_4(B_4)]$$

to define a linear extension of P. Technically speaking, we have not precisely defined a particular linear extension, since we intend that the choice of the extension on the blocks B_1 and B_3 is arbitrary.

In 1951, Hiraguchi [24] proved the following key lemma. A proof is provided here as the basic idea is essential for several of our proofs to follow.

Lemma 2.2.1. *If C is a chain in a poset P, then there exists a linear extension L of P so that $u > v$ in L for every $(u, v) \in \mathrm{Inc}(P)$ with $u \in C$.*

Proof. Label the points in the chain C as $u_1 < u_2 < \cdots < u_s$. Then partition the points in $P - C$ into blocks $B_1, B_2, \ldots, B_{s+1}$ where an element v from $P - C$ belongs to B_1 if $v \not> u_1$; otherwise v belongs to B_{j+1} where j is the largest integer so that $v > u_j$. Then set

$$L = [B_1 < u_1 < B_2 < u_2 < B_3 < \cdots < B_s < u_s < B_{s+1}].$$

It is clear that L satisfies the requirement of the lemma. □

Returning to the poset shown in Figure 5, we see that the previous discussion regarding this poset serves to illustrate the application of Lemma 2.2.1 to the chain $C = \{u_1, u_2, u_3\}$. However, as we will use this technique later, we elected to also specify how the linear extension L would order blocks B_2 and B_4.

It is useful to view the construction in Lemma 2.2.1 as pushing the chain C "up" while forcing all elements of $P - C$ "down" as low as possible relative to C, so we say that the resulting linear extension puts C "over" $P - C$. Also, there is a dual form of this lemma, i.e., there is a linear extension L with $u > v$ in L whenever $(u, v) \in \mathrm{Inc}(P)$ and $v \in C$. This extension puts $P - C$ "over" C.

Hiraguchi noted in [24] that when Lemma 2.2.1 is used for each chain in a minimum chain cover provided by Dilworth's theorem, there is a natural upper bound of dimension.

Lemma 2.2.2. *If P is a poset, then* $\dim(P) \leq \mathrm{width}(P)$.

The principal result in [24] is the following upper bound.

Theorem 2.2.3. [Hiraguchi's Inequality] *If P is a poset, then* $\dim(P) \leq |P|/2$, *when* $|P| \geq 4$.

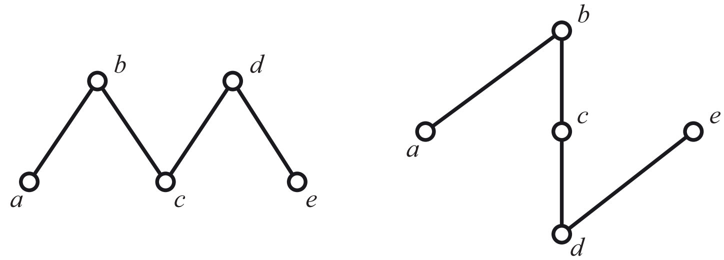

Figure 6: Two small indecomposable posets with width at least 3

Hiraguchi's original 1951 proof of this inequality was relatively complicated. In 1955, he gave an updated and somewhat streamlined proof [25], and in 1974, Bogart gave a more polished version [9]. Subsequently, Kimble [30] and Trotter [39] discovered the following inequality, and this result yields an elegant proof of Hiraguchi's theorem.

Theorem 2.2.4. *If A is an antichain in a poset P, then* $\dim(P) \leq \max\{2, |P - A|\}$.

Proof. It is straightforward to check that $\dim(P) \leq 2$ when $|P - A| \leq 1$. For the case $|P - A| = 2$. If the inequality fails, then there is a 3-irreducible (and therefore indecomposable) poset, having width at least 3 (by Lemma 2.2.2) and consisting of an antichain plus at most 2 other points. Note that such posets do not have any maximum or minimum points. There are only two (up to duality) posets satisfying these requirements. These two posets are shown in Figure 6. It is an easy exercise to show that both of these posets have dimension 2. On the other hand, this can also be done by checking the full list of 3-irreducible posets. Now assume the assertion is true for $|P - A| = k \geq 2$. For the case $|P - A| = k + 1$, let $Q = P - \{x\}$ where $x \in P - A$. Then $|Q - A| = k$ and by our assumption, we have $\dim(Q) \leq \max\{2, |Q - A|\}$. It follows that $\dim(P) \leq \dim(Q) + 1 \leq \max\{2, |Q - A|\} + 1 \leq \max\{2, |P - A|\}$ using Theorem 2.1.1. $\square$

2.3 The Removable Pair Conjecture

When $|P| \geq 3$, a distinct pair $\{x, y\}$ is called a *removable pair* if $\dim(P) \leq 1 + \dim(P - \{x, y\})$. When $\dim(P) = 3$, every distinct pair of points is a removable pair. (This can be verified by checking the full list of 3-irreducible posets in Figure 3 and Figure 4.) However, when $\dim(P) \geq 4$, no comparable pair in the standard example S_d is a removable pair. Also, among the incomparable pairs, only the critical pairs are removable.

Conjecture 2.3.1. [The Removable Pair Conjecture] *If P is a poset and $|P| \geq 3$, then P has a removable pair, i.e., there exist distinct elements x and y in P so that $\dim(P) \leq 1 + \dim(P - \{x, y\})$.*

Although its origins have been obscured with the passage of time, the Removable Pair Conjecture (RPC) has been investigated by researchers for more than 60 years. The first published reference to the RPC seems to be in a 1975 paper of Trotter [39]. We also note that the RPC was one of the "Unsolved Problems", assembled by the editorial board of *Order* and appearing for more than 10 years in each issue of the journal (see [28]).

If the RPC holds, then a simple inductive proof of Hiraguchi's theorem could be obtained just by establishing the base case: $\dim(P) \leq 2$ when $|P| \leq 5$. This is an easy exercise, as a counterexample would have to be an indecomposable poset with width at least 3. Same as before, there are only two such posets, the ones shown in Figure 6, and both have dimension 2.

Here are two of many conditions which guarantee that a pair is removable. The first result is due to Hiraguchi [24] while the second is part of the folklore of the subject, although it is implicit in Theorem 7.4 as presented in [42]. We provide a short proof since this result will be quite useful to us in proving our main theorems later.

Theorem 2.3.2. *Let P be a poset with $|P| \geq 3$, let a and b be distinct points in P with $a \in \mathrm{Min}(P)$ and $b \in \mathrm{Max}(P)$. If $a \parallel b$ in P, then $\{a, b\}$ is a removable pair.*

Proof. Let $Q = P - \{a, b\}$, let $t = \dim(Q)$ and let $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of Q. For each $i \in \{1, 2, \ldots, t\}$, let $R_i = [a < L_i < b]$. Then set

$$R_{t+1} = [D(b) < b < Q - D(b) - U(a) < a < U(a)].$$

Evidently, $\{R_1, R_2, \ldots, R_t, R_{t+1}\}$ is a realizer of P so that $\dim(P) \leq 1 + \dim(Q)$. $\square$

Theorem 2.3.3. *Let P be a poset with $|P| \geq 3$, let a and a' be distinct minimal elements in P. If $U(a) \subseteq U(a')$, then $\{a, a'\}$ is a removable pair.*

Proof. Let $Q = P - \{a, a'\}$, let $t = \dim(Q)$ and let $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of Q. For each $i \in \{1, 2, \ldots, t\}$, let $R_i = [a < a' < L_i]$. Then set

$$R_{t+1} = [Q - U(a') < a' < U(a') - U(a) < a < U(a)].$$

Evidently, $\{R_1, R_2, \ldots, R_t, R_{t+1}\}$ is a realizer of P so that $\dim(P) \leq 1 + \dim(Q)$. $\square$

Several strong versions of the RPC conjecture have been proposed, and some of these have been disproved. Bogart [8] suggested that a removable pair $\{x, y\}$ could always be found among the elements of $\mathrm{Max}(P) \cup \mathrm{Min}(P)$. This was disproved by Trotter and Monroe [44] who constructed for each $t \geq 1$, a poset P_t so that $|P| = (3t+1)^2 + (6t+2)$, $\dim(P) = 4t+2$, and $\dim(P - \{x, y\}) = 4t$ for every distinct pair $x, y \in \mathrm{Max}(P) \cup \mathrm{Min}(P)$. Subsequently, Bogart and Trotter conjectured [43] that every critical pair was removable, a result motivated by their work on interval dimension. However, this form of the RPC was disproved by Reuter [36]. Reuter's counterexample, shown in Figure 7, is a 4-dimensional poset P on 12 points. Note that (x, y) forms a critical pair in P whose removal will decrease the dimension of P by 2. i.e., $\dim(P - \{x, y\}) = 2$. Subsequently, Kierstead and Trotter [29] gave a general construction (also shown in Figure 7) which shows that for every $d \geq 5$, there is a poset P_d containing a critical pair (x, y) so that $\dim(P_d) = d$ and $\dim(P_d - \{x, y\}) = d - 2$.

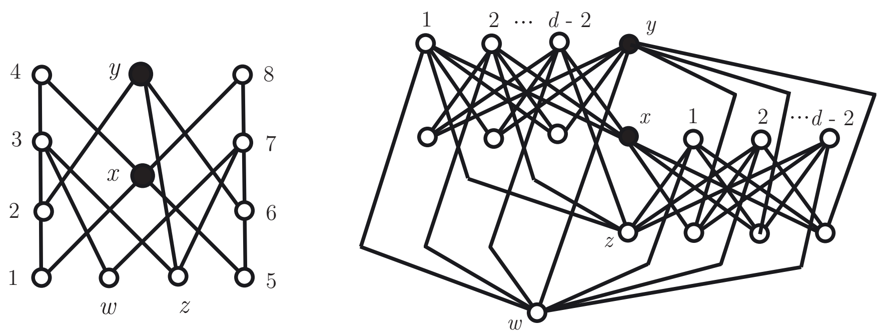

Figure 7: Critical pairs need not be removable

We note that it is still open to determine whether every poset P with $|P| \geq 3$ contains *some* critical pair which is removable.

Although the RPC remains open, there are some useful removal theorems which are more general. We say that two pairwise disjoint chains C and C' in a poset P are *incomparable* when $x \parallel y$ in P for every $x \in C$ and $y \in C'$. The following results are due to Hiraguchi [24]. The first is a straightforward generalization of Lemma 2.2.1 to two incomparable chains. The second is an immediate corollary of the first.

Lemma 2.3.4. *Let C_1 and C_2 be non-empty incomparable chains in a poset P. Then there exist linear extensions L and L' of P so that*

1. *C_1 is over $P - C_1$ and $P - C_2$ is over C_2 in L, and*
2. *C_2 is over $P - C_2$ and $P - C_1$ is over C_1 in L'.*

Lemma 2.3.5. *Let C_1 and C_2 be non-empty incomparable chains in a poset P. If there is some point of P which does not belong to either chain, then* $\dim(P) \leq 2 + \dim(P - (C_1 \cup C_2))$.

Combining Theorem 2.2.4 with Theorem 2.3.3 and Lemma 2.3.5, we have the following simple proof to Hiraguchi's inequality.

Proof of Theorem 2.2.3. Suppose the theorem is false and we choose a counterexample P with $|P|$ as small as possible. Then P is irreducible, and therefore indecomposable. Let $d = \dim(P)$. Since P is a counterexample, we have $d > |P|/2$. By appealing to the full list of 3-irreducible posets in Figure 3 and Figure 4, d is at least 4. Then Theorem 2.2.4 implies $|P|$ is at least 8.

Let a and a' be distinct minimal elements in P. If $U(a) \subseteq U(a')$, then by Theorem 2.3.3, we have

$$|P|/2 < d = \dim(P) \leq 1 + \dim(P - \{a, a'\}) \leq 1 + |P - \{a, a'\}|/2 = |P|/2.$$

Hence we have $U(a) \nsubseteq U(a')$. Similarly, we have $U(a') \nsubseteq U(a)$. So there exist $b, b' \in P$ such that $b \in U(a) - U(a')$ and $b' \in U(a') - U(a)$. Let $C_1 = \{a < b\}$ and $C_2 = \{a' < b'\}$ be two chains in P. Note that C_1 and C_2 are two incomparable chains. By Lemma 2.3.5, we have

$$|P|/2 < d = \dim(P) \leq 2 + \dim(P - (C_1 \cup C_2)) \leq 2 + |P - (C_1 \cup C_2)|/2 = |P|/2.$$

The contradiction completes the proof of Theorem 2.2.3. □

2.4 Statements of Main Theorems

Our principal theorem bounds the dimension of a poset P by the maximum size of matchings in comparability graph G_P and incomparability graph H_P.

Theorem 2.4.1. *Let P be a poset and let G_P and H_P be, respectively, the comparability graph and the incomparability graph of P. If $\dim(P) = d \geq 3$, then there is a matching of size d in G_P, and there is a matching of size d in H_P.*

As our primary focus is on the combinatorial properties of posets, we define a *chain matching* in a poset P as a family of pairwise disjoint 2-element chains in P. Dually, an *antichain matching* in P is a family of pairwise disjoint 2-element antichains in P. As the statements require separate proofs, we elect to restate Theorem 2.4.1 as two theorems, one for chain matchings and the other for antichain matchings.

Theorem 2.4.2. [Chain Matching Theorem] *Let P be a poset. If* $\dim(P) = d \geq 3$*, then P has a chain matching of size d.*

Theorem 2.4.3. [Antichain Matching Theorem] *Let P be a poset. If* $\dim(P) = d \geq 3$*, then P has an antichain matching of size d.*

Notice that these statements in Theorem 2.4.2 and Theorem 2.4.3 are not true when $d \leq 2$, as a 2-element antichain has a chain matching of size 0 and an antichain matching of size 1. For $d = 3$, it is straightforward to check that every 3-irreducible poset (see Figure 3 and 4) has both a chain matching of size 3 and an antichain matching of size 3. As a consequence, when presenting the proofs of two matching theorems, we will restrict our attention to posets with dimension at least 4.

Also note that Hiraguchi's inequality (Theorem 2.2.3) is an immediate corollary of either Theorem 2.4.2 or Theorem 2.4.3, as points in P which form a chain matching or an antichain matching are distinct.

We comment that there are other well-known instances of combinatorial problems with analogous statements for chains and antichains. For example, we have Dilworth's classic theorem [14] and the dual statement due to Mirsky [34].

Theorem 2.4.4. [Dilworth] *A poset of width w can be partitioned into w chains.*

Theorem 2.4.5. [Mirsky] *A poset of height h can be partitioned into h antichains.*

Second, we have the considerable strengthening of Dilworth's theorem due to Greene and Kleitman [23], with the dual result due to Greene [22].

Theorem 2.4.6. [Greene-Kleitman] *Let P be a poset. Then for every $k \geq 1$, there is a chain partition $P = C_1 \cup C_2 \cup \cdots \cup C_t$ so that for $i = k$ and $i = k+1$, the maximum size of a subposet Q of P with* $\text{height}(Q) \leq i$ *is* $\sum_{j=1}^{t} \min\{i, |C_j|\}$.

Theorem 2.4.7. [Greene] *Let P be a poset. Then for every $k \geq 1$, there is an antichain partition $P = A_1 \cup A_2 \cup \cdots \cup A_s$ so that for $i = k$ and $i = k+1$, the maximum size of a subposet Q of P with* $\text{width}(Q) \leq i$ *is* $\sum_{j=1}^{s} \min\{i, |A_j|\}$.

More recently, we have the following theorem proved by Duffus and Sands [15], with the dual result due to Howard and Trotter [26].

Theorem 2.4.8. [Duffus-Sands] *Let n and k be integers with $n \geq k \geq 3$, and let P be a poset. If $n \leq |C| \leq n + (n-k)/(k-2)$ for every maximal chain C in P, then P has k pairwise disjoint maximal antichains.*

Theorem 2.4.9. [Howard-Trotter] *Let n and k be integers with $n \geq k \geq 3$, and let P be a poset. If $n \leq |A| \leq n + (n-k)/(k-2)$ for every maximal antichain A in P, then P has k pairwise disjoint maximal chains.*

As is the case with Dilworth's theorem and its dual, the theorem for chain matchings is more challenging than the theorem for antichain matchings, and we do not know of any "perfect graph" underpinning that makes the two results equivalent. In section 2.6, we will also show that our chain matching theorem cannot be extended to cover graphs by showing that there is a d-dimensional poset for which the maximum size of a matching in the cover graph is $O(\log d)$.

2.5 Three New Inequalities for Dimension

In [39], Trotter proved the following inequality.

Theorem 2.5.1. *Let P be a poset which is not an antichain, and let $w = \mathrm{width}(P - \mathrm{Max}(P))$. Then $\dim(P) \leq w + 1$.*

This inequality is tight for all $w \geq 1$, and we refer the reader to [39] for details. In this same paper, Trotter also proved the following inequality.

Theorem 2.5.2. *Let A be a maximal antichain in a poset P which is not a chain, and let $w = \mathrm{width}(P - A)$. Then $\dim(P) \leq 2w + 1$.*

It is more complicated to show that the above inequality is tight, and the argument is presented in a separate paper [38]. In the context to follow, we will strengthen both results.

2.5.1 New Inequalities for Dimension

In order to prove our main theorems, we will need a straightforward extension of Theorem 2.5.2.

Theorem 2.5.3. *Let D be a non-empty down set in a poset P such that the up set $U = P - D$ is also non-empty. If $\dim(D) = t$ and $\mathrm{width}(U) = w$, then $\dim(P) \le t + w$.*

Proof. Since $\mathrm{width}(U) = w$, there is a partition of U into w chains. Let $U = C_1 \cup C_2 \cup \cdots \cup C_w$. Since $\dim(D) = t$, let $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of D. For each $i \in \{1, 2, \ldots, t\}$, let $R_i = [L_i < U]$. Then for each $j \in \{1, 2, \ldots, w\}$, let R_{t+j} be a linear extension which puts $P - C_j$ over C_j. Now we show $\{R_1, R_2, \ldots, R_{t+w}\}$ is a realizer of P. Let (x, y) be any incomparable pair in P. If $y \in C_j$, then $x > y$ in R_{t+j}. If $y \in D$ and $x \in U$, then $x > y$ in R_1. If $y \in D$ and $x \in D$, then there exists an i such that $x > y$ in L_i. Hence $x > y$ in R_i. Indeed, $\{R_1, R_2, \ldots, R_{t+w}\}$ is a realizer of P. $\square$

Note that the inequality in the preceding theorem is tight. Let P be the standard example $S_n (n \ge 3)$. Let $U \subseteq \mathrm{Max}(S_n)$ where $1 \le |U| \le n - 2$. Then U is an up set. Let $D = P - U$ be a down set. It is easy to see that $\dim(D) = n - |U| = n - \mathrm{width}(U)$. Hence $\dim(D) + \mathrm{width}(U) = n = \dim(P)$.

Of course, there is a dual version of Theorem 2.5.3 in which the roles of U and D are reversed. i.e., let U be a non-empty up set in a poset P such that the down set $D = P - U$ is also non-empty. If $\dim(U) = t$ and $\mathrm{width}(D) = w$, then $\dim(P) \le t + w$.

The next theorem provides a condition under which the inequality $\dim(P) \le 1 + \mathrm{width}(P - \mathrm{Max}(P))$ in Theorem 2.5.1 can be improved. Although this result is somewhat technical, it is an essential ingredient of proofs to follow.

Theorem 2.5.4. *Let P be a poset and let $Q = P - \mathrm{Max}(P)$. If $\mathrm{width}(Q) = w \ge 2$, and there is a point $x \in Q$ so that $\mathrm{width}(Q - \{x\}) = w - 1$, then $\dim(P) \le w$.*

Proof. We argue by contradiction and assume that P is a counterexample of minimum size. Then P is $(w+1)$-irreducible.

Let $C_1 \cup C_2 \cup \cdots \cup C_{w-1} \cup \{x\}$ be a chain cover of $Q = P - \mathrm{Max}(P)$. Setting $C_w = \{x\}$, we apply Lemma 2.2.1 to choose for each $i \in \{1, 2, \ldots, w\}$, a linear extension L_i which puts C_i over $P - C_i$. We then modify L_{w-1} and L_w to form L'_{w-1} and L'_w and show that the family $\mathcal{F}' = \{L_1, L_2, \ldots, L_{w-2}, L'_{w-1}, L'_w\}$ must be a realizer of P.

Let $C_{w-1} = \{u_1 < u_2 < \cdots < u_s\}$. The point x must be incomparable with at least one point of C_{w-1}; else $C_{w-1} \cup \{x\}$ is a chain and $\mathrm{width}(Q) \leq w-1$.

Then let $A_1 = \mathrm{Max}(P) \cap U(x)$ and $A_2 = \mathrm{Max}(P) - A_1$. Then set

$$L'_{w-1} = [L_{w-1}(P - A_2) < L^*_{w-1}(A_2)].$$

Note that since $A_2 \subseteq \mathrm{Max}(P)$, L'_{w-1} is well defined. Now in forming the linear extension L_w, there are only two blocks, and to distinguish these from the blocks appearing in L_{w-1}, we denote these as S_1 and S_2. Let $S_1 = P - U(x) - \{x\}$, and $S_2 = U(x)$. Now set

$$L'_w = [L_{w-1}(S_1) < x < L_{w-1}(S_2 - A_1) < L^*_{w-1}(A_1)].$$

By our assumption, $\dim(P) = w+1$. The family $\mathcal{F}' = \{L_1, L_2, \ldots, L_{w-2}, L'_{w-1}, L'_w\}$ cannot be a realizer of P. Hence there is some critical pair $(u, v) \in \mathrm{Crit}(P)$ with $u < v$ in each of the w linear extensions in $\mathcal{F}'$. Clearly, there is no i with $1 \leq i < w-2$ for which $u \in C_i$, otherwise we have $u > v$ in L_i. Now suppose that $u \in C_{w-1}$. Since $u > v$ in L_{w-1} and $u < v$ in L'_{w-1}, we must have $v \in A_2$. This implies that $v \parallel x$ in P, so that v belongs to block S_1 in L_w. If $u \in S_2$, then $u > x > v$ in L'_w. If $u \in S_1$, then $u > v$ in L_{w-1} implies $u > v$ in L'_w.

If $u = x$, then $u \parallel v$ implies $v \in S_1$. It follows that $u > v$ in L'_w.

Finally, suppose that $u \in \mathrm{Max}(P)$. Since $(u, v) \in \mathrm{Crit}(P)$, we also have $v \in \mathrm{Max}(P)$. First we assume $u \in A_1$. $u < v$ in L'_w implies $v \in A_1$ and $u > v$ in L_{w-1}.

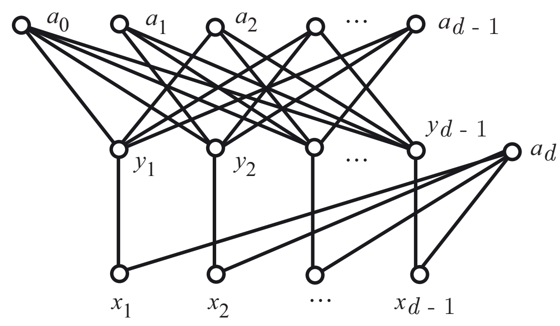

Figure 8: Posets witness that the inequality in Theorem 2.5.4 is tight

Hence we have $u > v$ in L'_{w-1}. So we must have $u \in A_2$. $u < v$ in L'_{w-1} implies $v \in A_2$ and $u > v$ in L_{w-1}. Note $A_2 \subseteq S_1$, it follows that $u > v$ in L'_w. The contradiction completes the proof of Theorem 2.5.4. $\square$

2.5.2 The Inequality of Theorem 2.5.4 is Tight

We show in Figure 8 a family of posets $\{P_d : d \geq 3\}$. Observe that the following properties hold:

1. $\mathrm{Max}(P_d) = \{a_0, a_1, a_2, \ldots, a_d\}$.

2. $\mathrm{width}(P_d - \mathrm{Max}(P_d)) = d - 1$.

3. $\mathcal{M} = \{\{x_i < y_i\} : 1 \leq i \leq d-1\}$ is a chain matching in P, but it is not maximum.

We now show that $\dim(P_d) = d$ for all $d \geq 3$. Accordingly, this shows that the inequality $\dim(P) \leq 1 + \mathrm{width}(P - \mathrm{Max}(P))$ is best possible. However, it also shows that the condition of Theorem 2.5.4 cannot be weakened, since the chains covering $P_d - \mathrm{Max}(P_d)$ have size 2.

Suppose to the contrary that $\dim(P) \leq d - 1$. We note that the subposet of P_d determined by the points in $\{x_i : 1 \leq i \leq d-1\} \cup \{a_i : 1 \leq i \leq d-1\}$ is isomorphic to the standard example S_{d-1}, so we must then have $\dim(P_d) = d - 1$. Now let

$\mathcal{F} = \{L_1, L_2, \ldots, L_{d-1}\}$ be a realizer of P_d. We may assume that the linear extensions in $\mathcal{F}$ have been labeled so that $x_i > a_i$ in L_i for each $i \in \{1, 2, \ldots, d-1\}$. Then for all $i, j \in \{1, 2, \ldots, d-1\}$ with $i \neq j$, we have

$$x_j < y_j < a_i < x_i < y_i < a_0 \quad \text{in } L_i.$$

Since $a_d \parallel y_j$ for $j \in \{1, 2, \ldots, d-1\}$ and $a_d > x_i$ for each $i \in \{1, 2, \ldots, d-1\}$, we must therefore have $x_i < a_d < y_i < a_0$ in L_i for each $i \in \{1, 2, \ldots, d-1\}$. This is a contradiction since $\mathcal{F}$ does not reverse the incomparable pair (a_d, a_0).

Note that P_3 is not irreducible, as $P_3 - \{a_1, a_2\}$ is isomorphic to the dual of the poset D shown in Figure 4. This poset is traditionally called the "chevron". However, for each $d \geq 4$, the poset P_d is d-irreducible.

2.5.3 An Inequality Involving Matchings

Here is another result which gives a condition under which the the inequality $\dim(P) \leq 1 + \text{width}(P - \text{Max}(P))$ of Theorem 2.5.1 can be tightened.

When $\mathcal{M}$ is a maximum chain matching in a poset P, we let $A(\mathcal{M})$ denote the set of all points in P which are not covered by the chains in $\mathcal{M}$. Evidently, $A(\mathcal{M})$ is an antichain.

Theorem 2.5.5. *Let P be a poset which is not an antichain, and let $\mathcal{M}$ be a maximum chain matching in P. If $\mathcal{M}$ has size m and $A(\mathcal{M}) \subseteq \text{Max}(P)$ (or $A(\mathcal{M}) \subseteq \text{Min}(P)$), then $\dim(P) \leq \max\{2, m\}$.*

Proof. We prove the case when $A(\mathcal{M}) \subseteq \text{Max}(P)$, the other case of $A(\mathcal{M}) \subseteq \text{Min}(P)$ is dual. We will argue by contradiction. We assume the conclusion of the theorem is false and choose a counterexample P with $|P|$ minimum. Let $d = \dim(P)$, noting that $d \geq 3$. Set $m = |\mathcal{M}|$. Since P is not an antichain, $m \geq 1$.

Let $Q = P - \text{Max}(P)$. Then $1 \leq \text{width}(Q) \leq m < d$. By Theorem 2.5.1, we conclude $m = \text{width}(Q) = d - 1$, which forces $m \geq 2$.

Claim 1. $A(\mathcal{M}) = \mathrm{Max}(P)$.

Proof. Suppose to the contrary that there is a maximal element y so that y belongs to a chain C in the matching $\mathcal{M}$. Let x be the other point in C. It follows that $P - \mathrm{Max}(P)$ is covered by m chains, with one of the chains being the single point $\{x\}$. Now Theorem 2.5.4 implies that $\dim(P) \leq m$. The contradiction completes the proof of the Claim 1. $\square$

Claim 2. $\dim(P - \{u\}) = d - 1$ *for every* $u \in \mathrm{Max}(P)$.

Proof. Let $u \in \mathrm{Max}(P)$ and assume that $\dim(P - \{u\}) = d$. By Claim 1, u does not belong to any chains in $\mathcal{M}$, therefore $\mathcal{M}$ is also a maximum chain matching in $P - \{u\}$. It follows that $P - \{u\}$ would be a counterexample, contradicting our choice of P. The contradiction completes the proof of Claim 2. $\square$

Label the chains in $\mathcal{M}$ as $\{C_i = \{x_i < y_i\} : 1 \leq i \leq m\}$. Apply Lemma 2.2.1 for each chain C_i to obtain a linear extension L_i which puts C_i over $P - C_i$. Note that in the construction of L_i, there are three blocks which we denote $B_{i,1}$, $B_{i,2}$ and $B_{i,3}$ so that $L_i = [B_{i,1} < x_i < B_{i,2} < y_i < B_{i,3}]$. For each $i \in \{1, 2, \ldots, m-1\}$, we then modify L_i to form L'_i by altering the order on the three blocks as follows:

$$L'_i(B_{i,j}) = [B_{i,j} - \mathrm{Max}(P) < L^*_m(B_{i,j} \cap \mathrm{Max}(P))] \quad \text{for } j \in \{1, 2, 3\}.$$

Finally, we modify L_m to form L'_m by setting:

$$L'_m(B_{m,j}) = [B_{m,j} - \mathrm{Max}(P) < L_m(B_{m,j} \cap \mathrm{Max}(P))] \quad \text{for } j \in \{1, 2, 3\}.$$

Note that elements always remain in the same block. We simply pull the maximal elements to the top of the block and then order them in a manner that is dual to how they are ordered in L'_m.

Since P is a counterexample, the family $\mathcal{F} = \{L'_i : 1 \leq i \leq m\}$ cannot be a realizer of P, so there is a critical pair $(u, v) \in \mathrm{Crit}(P)$ for which $u < v$ in L'_i for all $i \in \{1, 2, \ldots, m\}$. Since L_i puts C_i over $P - C_i$, so does L_i'. If $u \in C_i$ for some $i \in \{1, 2, \ldots, m\}$, then $u > v$ in L_i'. Hence we have $u \in \text{Max}(P)$, and in turn this forces $v \in \text{Max}(P)$.

Since $\dim(P - \{u\}) = d - 1$ by Claim 2, u cannot be a loose point of P. We know that the maximal element u is not a minimal element of P, so $u > x_i$ in P for all $i \in \{1, 2, \ldots, m\}$. Since $(u, v) \in \text{Crit}(P)$, we know that $v > x_i$ for all $i \in \{1, 2, \ldots, m\}$.

For every $i \in \{1, 2, \ldots, m\}$, since $u, v > x_i$ in P, we know that u, v belong to one of $B_{i,2}$ and $B_{i,3}$. However, if u and v are in the same block in each L_i', then in L_1' they are ordered in reverse order to how they are ordered in L_m'. So there must exist a $j \in \{1, 2, ..., m\}$ such that we have $u \in B_{j,2}$ and $v \in B_{j,3}$. This implies that $v > y_j$. However, this implies that the chain matching $\mathcal{M}$ is not maximum, since we could replace $C_j = \{x_j < y_j\}$ with $C' = \{x_j < u\}$ and $C'' = \{y_j < v\}$. The contradiction completes the proof of the theorem. $\square$

Again, the poset P_d illustrated in Figure 8 shows that the inequality in Theorem 2.5.5 is tight. Here $dim(P_d) = d$, $m = d$ and $\mathcal{M} = \{\{x_i < y_i\} : 1 \leq i \leq d-2\} \cup \{x_{d-1} < a_d\} \cup \{y_{d-1} < a_0\}$.

2.6 Chain Matchings

Let $\mathcal{M}$ be a maximum chain matching in a poset P. As in the preceding section, we let $A(\mathcal{M})$ denote the antichain consisting of the points in P which are not covered by chains in $\mathcal{M}$. If $A(\mathcal{M}) = \emptyset$, then $\dim(P) \leq \text{width}(P) \leq m$, so our focus in this section will be on posets for which $A(\mathcal{M})$ is a non-empty antichain. In this case, we then let $\mathbb{U}(\mathcal{M})$ denote the set of all chains $C \in \mathcal{M}$ for which there is an element $a \in A(\mathcal{M})$, an integer $s \geq 1$ and a sequence $\{C_i = \{x_i < y_i\} : 1 \leq i \leq s\}$ of distinct chains in $\mathcal{M}$ so that (1) $y_1 > a$ in P, (2) $y_{i+1} > x_i$ in P whenever $1 \leq i < s$ and (3) $C = C_s$. Dually, $\mathbb{D}(\mathcal{M})$ denotes the set of all chains $C' \in \mathcal{M}$ for which there is an element $a' \in A(\mathcal{M})$, an integer $s' \geq 1$ and a sequence $\{C_i' = \{x_i' < y_i'\} : 1 \leq i \leq s'\}$ of

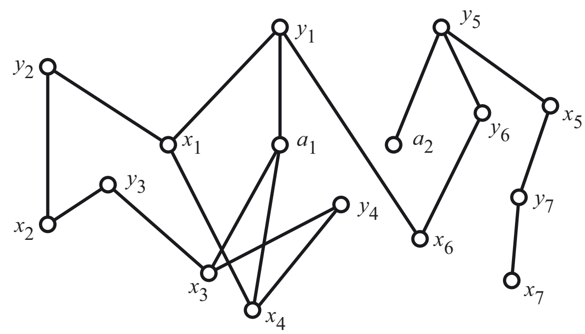

Figure 9: Characterizing chains in a maximum matching

distinct chains in $\mathcal{M}$ so that (1) $x'_1 < a'$ in P, (2) $x'_{i+1} < y'_i$ in P whenever $1 \leq i < s'$ and (3) $C' = C_{s'}$.

In Figure 9, we show a poset P and a maximum chain matching $\mathcal{M} = \{\{x_i < y_i\} : 1 \leq i \leq 7\}$. It is easy to see that:

1. The chains $\{x_1 < y_1\}$, $\{x_2 < y_2\}$, $\{x_3 < y_3\}$ and $\{x_4 < y_4\}$ belong to $\mathbb{U}(\mathcal{M}) \cap \mathbb{D}(\mathcal{M})$.

2. The chain $\{x_5 < y_5\}$ belongs to $\mathbb{U}(\mathcal{M})$ but not to $\mathbb{D}(\mathcal{M})$.

3. The chain $\{x_6 < y_6\}$ belongs to $\mathbb{D}(\mathcal{M})$ but not to $\mathbb{U}(\mathcal{M})$.

4. The chain $\{x_7 < y_7\}$ does not belong to $\mathbb{U}(\mathcal{M})$ or to $\mathbb{D}(\mathcal{M})$.

2.6.1 Existence of Pure Maximum Chain Matchings

We say that a maximum chain matching $\mathcal{M}$ in P is *pure* if $\mathbb{U}(\mathcal{M}) \cap \mathbb{D}(\mathcal{M}) = \emptyset$. A maximum chain matching $\mathcal{M}$ with $A(\mathcal{M}) \subseteq \mathrm{Max}(P)$ is pure since $\mathbb{U}(\mathcal{M}) = \emptyset$. Dually, a maximum chain matching $\mathcal{M}$ with $A(\mathcal{M}) \subseteq \mathrm{Min}(P)$ is pure. On the other hand, there are posets having no maximum chain matching $\mathcal{M}$ for which $A(\mathcal{M})$ is a subset of $\mathrm{Max}(P)$ or $\mathrm{Min}(P)$. Nevertheless, all posets have pure maximum chain matchings, as evidenced by the following Lemma, first exploited by Trotter in [40]. For the sake of completeness, the elementary argument for this fact is included here.

Lemma 2.6.1. *Every poset has a pure maximum chain matching.*

Proof. Let P be a poset. Choose a maximum chain matching $\mathcal{M}$ in P which maximizes the quantity $q(\mathcal{M})$ defined by:

$$q(\mathcal{M}) = \sum_{a \in A(\mathcal{M})} |D(a)| \tag{1}$$

We claim that $\mathcal{M}$ is pure. To see this, suppose that there is a chain $C \in \mathbb{U}(\mathcal{M}) \cap \mathbb{D}(\mathcal{M})$. Then after a suitable relabeling of the chains in $\mathcal{M}$, there are elements $a, a' \in A(\mathcal{M})$ (not necessarily distinct), a positive integer r and a sequence $\{C_i = \{x_i < y_i\} : 1 \leq i \leq r\}$ of distinct chains from $\mathcal{M}$ so that $y_1 > a$ in P, $x_r < a'$ in P and $y_{i+1} > x_i$ in P whenever $1 \leq i < r$.

If $a \neq a'$, we could remove the r chains in the sequence from $\mathcal{M}$ and replace them by the following set of $r+1$ chains:

$$\{a < y_1\}, \{x_r < a'\}, \{\{x_i < y_{i+1}\} : 1 \leq i < r\}$$

This would contradict the assumption that $\mathcal{M}$ is a maximum chain matching in P. We conclude that $a = a'$. In this case, we form a maximum chain matching $\mathcal{M}'$ from $\mathcal{M}$ by replacing the chains in the sequence by the following set:

$$\{x_r < a\}, \{\{x_i < y_{i+1}\} : 1 \leq i < r\}$$

Now the antichain $A(\mathcal{M}')$ is obtained from $A(\mathcal{M})$ by replacing a by y_1. Since $a < y_1$, we conclude that $q(\mathcal{M}') \geq q(\mathcal{M}) + 1$. The contradiction completes the proof of the lemma. □

Let $\mathcal{M}$ be any pure maximum chain matching in P. Then let U be the subposet of P consisting of those points in P covered by chains in $\mathbb{U}(\mathcal{M})$. Let $D = P - U$.

Lemma 2.6.2. *U is an up set in P, D is a down set in P.*

Proof. Fix an arbitrary point $a \in U$. For any point b with $b > a$ in P, we show that $b \in U$. Let $C_a = \{x_a < y_a\}$ be the chain matching in $\mathcal{M}$ which covers a. If $b \in A(\mathcal{M})$, then $C_a \in \mathbb{D}(\mathcal{M})$, which contradicts the fact that $\mathcal{M}$ is pure. Hence b is covered by chains in $\mathcal{M}$. Let $C_b = \{x_b < y_b\}$ be the chain matching in $\mathcal{M}$ which covers b. Since $b > a$, we must have $y_b \geq b > a \geq x_a$. Then $C_a \in \mathbb{U}(\mathcal{M})$ implies $C_b \in \mathbb{U}(\mathcal{M})$ and $b \in U$. Hence U is an up set, $D = P - U$ is a down set. □

2.6.2 The Proof of the Chain Matching Theorem

We are now ready to prove that a poset P with $\dim(P) = d \geq 4$ has a chain matching of size d. We argue by contradiction. Suppose this assertion is false and choose a counterexample P with $|P|$ minimum. Let u be any element of P. If $\dim(P - \{u\}) = \dim(P)$, then $P - \{u\}$ is also a counterexample, contradicting our choice of P. We conclude that $\dim(P - \{u\}) = \dim(P) - 1$. Since u is arbitrary, it follows that P is d-irreducible.

Let m be the maximum size of a chain matching in P. If $m \leq d - 2$, then $P - \{u\}$ is also a counterexample for any $u \in P$. Hence $m = d - 1$. Also, we note that P^*, the dual of P, is also a counterexample of minimum size. This observation allows us to take advantage of duality in the arguments to follow.

Again, let $\mathcal{M}$ be any pure maximum chain matching in P. If $A(\mathcal{M}) \subseteq \mathrm{Max}(P)$, then Theorem 2.5.5 implies that $\dim(P) \leq m$. The contradiction shows that $A(\mathcal{M}) \nsubseteq \mathrm{Max}(P)$. By duality, we also know that $A(\mathcal{M}) \nsubseteq \mathrm{Min}(P)$. This implies that $\mathbb{U}(\mathcal{M}) \neq \emptyset \neq \mathbb{D}(\mathcal{M})$. Set

$$\mathbb{I}(\mathcal{M}) = \mathcal{M} - (\mathbb{D}(\mathcal{M}) \cup \mathbb{U}(\mathcal{M})).$$

Taking advantage of duality, we can assume that $|\mathbb{U}(\mathcal{M})| \leq |\mathbb{D}(\mathcal{M})|$. Since $m = d - 1 \geq 3$, it follows that $m_0 = |\mathbb{D}(\mathcal{M}) \cup \mathbb{I}(\mathcal{M})| \geq 2$. Then let U be the subposet of P consisting of those points of P covered by chains in $\mathbb{U}(\mathcal{M})$. Let $D = P - U$. By Lemma 2.6.2, U is an up set in P, D is a down set. Since $\mathbb{U}(\mathcal{M}) \neq \emptyset \neq \mathbb{D}(\mathcal{M})$, both

U and D are non-empty. Furthermore, the maximum size of a chain matching in D is m_0. Also, the chains in $\mathbb{D}(\mathcal{M}) \cup \mathbb{I}(\mathcal{M})$ cover all points of D except the points in $A(\mathcal{M})$ which belong to $\mathrm{Max}(D)$. Then $\dim(D) \leq m_0$ by Theorem 2.5.5.

On the other hand, the $m - m_0$ chains in $\mathbb{U}(\mathcal{M})$ cover U, so $\mathrm{width}(U) \leq m - m_0$. Now Theorem 2.5.3 implies that $\dim(P) \leq \dim(D) + \mathrm{width}(U) \leq m_0 + (m - m_0) = m$. The contradiction completes the proof.

2.6.3 Chain matching theorem of 3-dimensional posets

It is possible to prove that a 3-dimensional poset has a chain matching of size 3 without using the knowledge about the full list of 3-irreducible posets. Here we need another elementary result due to Baker, Fishburn and Roberts [3].

Theorem 2.6.3. *If P is a planar poset and P has a unique minimal element and a unique maximal element, then* $\dim(P) \leq 2$.

Traditionally, a unique minimal element of a poset is called a *zero*, while a unique maximal element is called a *one*. Accordingly, the preceding result is usually phrased as asserting that a planar poset with a zero and a one has dimension at most 2.

Now we are ready to prove the chain matching theorem for the case $d = 3$, without appealing to the list of 3-irreducible posets.

Theorem 2.6.4. *Let P be a poset. If* $\dim(P) = 3$*, then P has a chain matching of size* 3.

Proof. We argue by contradiction. We assume the conclusion of the above theorem is false and choose a counterexample P with $|P|$ minimum. Then P is irreducible, therefore indecomposable. Note that $\mathrm{width}(P) \geq \dim(P) = 3$. By Theorem 2.2.4, we have $|P| \geq 3 + \mathrm{width}(P) \geq 6$. Let $\mathcal{M}$ of size m be any pure maximum chain matching in P. Since P is a counterexample, m is either 1 or 2. If $A(\mathcal{M}) \subseteq \mathrm{Max}(P)$ or $A(\mathcal{M}) \subseteq \mathrm{Min}(P)$, then Theorem 2.5.5 implies $\dim(P) \leq \max\{2, m\} = 2$. The contradiction implies $A(\mathcal{M}) \not\subseteq \mathrm{Max}(P)$ and $A(\mathcal{M}) \not\subseteq \mathrm{Min}(P)$. It then implies $\mathbb{U}(\mathcal{M}) \neq \emptyset \neq \mathbb{D}(\mathcal{M})$. As $m \leq 2$, we conclude that $|\mathbb{U}(\mathcal{M})| = |\mathbb{D}(\mathcal{M})| = 1$.

Let $\mathbb{U}(\mathcal{M}) = \{a < b\}$, $\mathbb{D}(\mathcal{M}) = \{c < d\}$. Since $|P| \geq 6$, $|A(\mathcal{M})| \geq 2$. There exist $x, y \in A(\mathcal{M})$ such that $x < b$ and $y > c$.

Case 1. $x = y$.

As $|P| \geq 6$, there exists $z \in A(\mathcal{M})$ and $z \neq x$. Note that z is incomparable to a, x and d. It follows that either $z < b$ and $z \parallel c$, or $z \parallel b$ and $z > c$, or $c < z < b$. Without loss of generality, we may assume $z < b$. Since $\{z < b\}, \{c < d\}$ is a chain matching of size 2, we have $a \parallel x$. Also $\{z < b\}, \{c < x\}$ is a chain matching of size 2 implies $a \parallel d$.

If $x \parallel d$, then $\{a, d, x, z\}$ is an antichain. Furthermore, no elements in P, except b and c, are comparable to any elements in $\{a, d, x, z\}$. Therefore, $P - \{b, c\}$ is an antichain. It then follow from Theorem 2.2.4 that $\dim(P) \leq 2$. The contradiction implies $x > d$. Note that $\{z < b\}, \{d < x\}$ is a chain matching of size 2 implies $a \parallel c$. Similarly we have $z \parallel c$. Note that no elements in P, except b, are comparable to a or z. Now $\{a, z\}$ is an autonomous subposet of P, which contradicts the fact that P is indecomposable.

Case 2. $x \neq y$.

As $x \neq y$, we may assume $x \parallel c$ and $y \parallel b$. Note that $\mathcal{M}$ is a pure matching, therefore $b \in \mathrm{Max}(P)$ and $c \in \mathrm{Min}(P)$. Since $\{x < b\}, \{c < y\}$ is a chain matching of size 2, we have $a \parallel d$. Also $\{x < b\}, \{c < d\}$ is a chain matching of size 2 implies $a \parallel y$. Similarly, we have $x \parallel y, d$. Note that no elements in P, except b and c, are comparable to any elements in $\{a, d, x, y\}$. As $\{a, x\}$ is not an autonomous subposet of P, we must have $a > c$. Dually, we have $d < b$.

If $|P| = 6$ (Figure 10), then we can add a "zero" and a "one" to P so that the resulting poset is still planar. It follows that $\dim(P) \leq 2$ from Theorem 2.6.3. Otherwise, $|P| \geq 7$, and there exists $z \in A(\mathcal{M})$ such that $z \neq x, y$. Note that z is

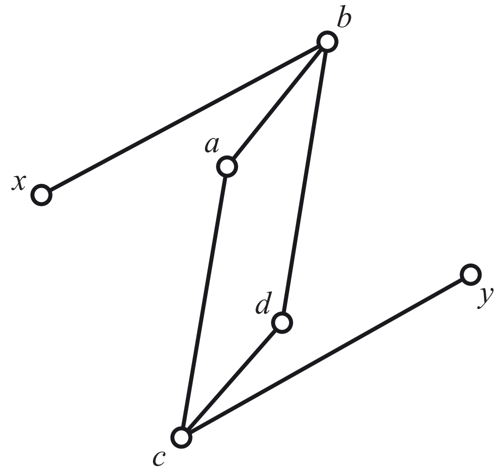

Figure 10: The dimension of P is at most 2

incomparable to a, d, x and y. It follows that either $z < b$ and $z \parallel c$, or $z \parallel b$ and $z > c$, or $c < z < b$. Without loss of generality, we may assume $z < b$. However, if $z \parallel c$, then $\{z, x\}$ is an autonomous subposet of P; if $z > c$, then $\{z, a\}$ is an autonomous subposet of P. The contradiction completes the proof of Theorem 2.6.4. □

2.6.4 Matchings in Cover Graphs

It is tempting to believe that our chain matching theorem can be strengthened by requiring that the chains in the matching be covers. In this setting, a chain matching $\mathcal{M}$ can be viewed as a matching in the cover graph. However, we will now show that no such extension is possible.

Let $k \geq 2$, and set $d = \binom{2k}{k}$. We construct a height 3 poset P_k which contains the standard example S_d as a subposet, yet the largest matching in the cover graph of P_k has size $2k$. The minimal elements of P_k are labeled as $a(S)$ where S is a k-element subset of $\{1, 2, \ldots, 2k\}$. Similarly, the maximal elements of P_k are labeled as $b(T)$ where T is a k-element subset of $\{1, 2, \ldots, 2k\}$. There are $2k$ other elements of P_k and these are labeled as $m_1, m_2, \ldots, m_{2k}$. For each k-element subset S of $\{1, 2, \ldots, 2k\}$ and each integer i with $1 \leq i \leq 2k$, we have $a(S) < m_i$ and $m_i < b(S)$ if and only if $i \in S$. It follows that that $a(S) < b(T)$ unless S and T are complementary subsets of

$\{1, 2, \ldots, 2k\}$. This shows that P_k contains the standard example S_d as a subposet. On the other hand, there are no covers between minimal and maximal elements, so a matching in the cover graph of P_k has size at most $2k$.

Inverting parameters, it is clear that for large d, there is a poset P for which the dimension of P is d, yet the largest matching in the cover graph of P has size $O(\log d)$. The next theorem says that this bound is tight up to a multiplicative constant.

Theorem 2.6.5. *For every $m \geq 1$, if P is a poset and the maximum size of a matching in the cover graph of P is m, then the dimension of P is at most* $(5^m+2m)/2$.

Proof. We argue by contradiction. Suppose the inequality fails for some $m \geq 1$ and let P be a counterexample of minimum size. Then let $\mathcal{M}$ be a maximum matching in the cover graph of P. We label the edges in $\mathcal{M}$ as $\{C_i = \{x_i < y_i\} : 1 \leq i \leq m\}$. Now y_i covers x_i in P for all $i \in \{1, 2, \ldots, m\}$.

Let $X(\mathcal{M})$ denote the elements of P which are not covered by edges in $\mathcal{M}$. Unlike the case for matchings in the comparability graph, we no longer know that $X(\mathcal{M})$ is an antichain. Nevertheless, we know that for each $x \in X(\mathcal{M})$, any edge in the cover graph incident with x have its other end point in one of the covers in $\mathcal{M}$. It follows that for each $x \in X(\mathcal{M})$ and each $i \in \{1, 2, \ldots, m\}$, there are five possibilities:

1. There are no edges in the cover graph with x as one endpoint and the other in C_i.

2. x covers y_i.

3. x covers x_i.

4. y_i covers x.

5. x_i covers x.

Accordingly, we may assign to each point $x \in X(\mathcal{M})$ a vector of length m with coordinate i being an integer from $\{1, 2, 3, 4, 5\}$ reflecting which of these possibilities holds for x and C_i.

Now suppose that x and y are distinct elements of $X(\mathcal{M})$ and they are both assigned to the same vector. Clearly, this implies that the subposet $Y = \{x, y\}$ is an autonomous subset in P. It follows that $\dim(P - \{x\}) = \dim(P)$. However, since $D(x) = D(y)$ and $U(x) = U(y)$, the cover graph of the subposet $P - \{x\}$ is obtained from the cover graph of P just by deleting x and the edges incident with x. It follows that $P - \{x\}$ is a counterexample of smaller size.

The contradiction shows that distinct points of $X(\mathcal{M})$ must be assigned to distinct vectors, so $|X(\mathcal{M})| \leq 5^m$. So altogether, P has at most $5^m + 2m$ points, and since $5^m + 2m \geq 7$, the conclusion of the theorem holds by appealing to Hiraguchi's inequality (Theorem 2.2.3). $\square$

2.7 Antichain Matchings

2.7.1 The Proof of the Chain Matching Theorem

We are now ready to present the proof of our main theorem for antichain matchings by showing that if $\dim(P) = d \geq 4$, then P has an antichain matching of size d. Again, we proceed by contradiction and choose a counterexample P of minimum size. As in the preceding section, we use our knowledge concerning the list of 3-irreducible posets and note that P must therefore be d-irreducible for some $d \geq 4$. Furthermore, the maximum size m of an antichain matching in P is $d - 1$.

Claim 1. *There do not exist distinct minimal elements a and a' in P with $U(a) \subseteq U(a')$.*

Proof. If such a pair could be found in P, set $Q = P - \{a, a'\}$ and note that $\dim(Q) = d - 1 \geq 3$ by Theorem 2.3.3. This implies that there is an antichain matching $\mathcal{M}$ of size $d - 1$ in Q. It follows that $\{a, a'\} \cup \mathcal{M}$ is an antichain matching of size d in P. The contradiction completes the proof of Claim 1.

Claim 2. $d \geq 5$.

Proof. Suppose to the contrary that $\dim(P) = 4$. Then $\text{width}(P) \geq \dim(P) = 4$. We consider a maximum antichain A in P. Suppose first that $A \subseteq \text{Min}(P)$. Choose four distinct elements of A and label them as a_1, a_2, a_3 and a_4. By Claim 1, we may choose elements x_1 and x_2 from $U(A)$ so that $x_1 > a_1$, $x_2 > a_2$, $x_1 \parallel a_2$ and $x_2 \parallel a_1$.

Since $\{a_1, a_2\}$, $\{x_1, x_2\}$ and $\{a_3, a_4\}$ is an antichain matching of size 3, we know that $C = P - (A \cup \{x_1, x_2\})$ is a chain. But then $P - A$ can be covered by three chains $\{x_1\}$, $\{x_2\}$ and C, and one of them (actually two of them) consists of a single point. It follows from Theorem 2.5.4 that $\dim(P) \leq 3$.

Hence $A \not\subseteq \text{Min}(P)$. Using duality, we may assume that $U(A) \neq \emptyset \neq D(A)$. Since $|A| \geq 4$, it follows that at least one of $U(A)$ or $D(A)$ is a chain. Without loss of generality, we assume $D(A)$ is a chain. Let u_1 be the least element of $D(A)$. Then choose $a_1 \in A$ with $u_1 < a_1$. Since P is irreducible, u_1 cannot be the only minimal element of P. It follows that there exists an element $a_2 \in A$ with $a_2 \in \text{Min}(P)$. Then by Claim 1, there is an element $x_2 \in U(A)$ with $x_2 > a_2$ and $x_2 \parallel u_1$. Since $\{x_2, u_1\}$, $\{a_1, a_2\}$ are antichains and $|A| \geq 4$, it follows that $U(A) - \{x_2\}$ is a chain. So $U(A)$ is covered by two chains, and one of them has size 1. Now Theorem 2.5.4 implies that $\dim(P - D(A)) \leq 2$. In turn, Theorem 2.5.3 implies that $\dim(P) \leq 3$. The contradiction completes the proof of Claim 2.

We now know that $\dim(P) \geq 5$. Choose distinct minimal elements a_1 and a_2 in P. By Claim 1, there are points x_1 and x_2 so that $x_1 > a_1$, $x_1 \parallel a_2$, $x_2 > a_2$ and $x_2 \parallel a_1$. Then $C_1 = \{a_1 < x_1\}$ and $C_2 = \{a_2 < x_2\}$ are disjoint incomparable chains. Set $Q = P - (C_1 \cup C_2)$. Then Lemma 2.3.5 implies that $\dim(Q) \geq d - 2 \geq 3$, so Q has an antichain matching $\mathcal{M}$ of size $d - 2$. It follows that $\mathcal{M} \cup \{\{a_1, a_2\}, \{x_1, x_2\}\}$ is an antichain matching of size d in P. The contradiction completes the proof of our antichain matching theorem.

2.7.2 Antichain matching theorem of 3-dimensional posets

Similar to the chain matching theorem, our antichain matching theorem for the case $d = 3$ can be proved directly, without appealing to the list of 3-irreducible posets.

Theorem 2.7.1. *Let P be a poset. If $\dim(P) = 3$, then P has an antichain matching of size* 3.

Proof. We argue by contradiction. We assume the conclusion of the above theorem is false and choose a counterexample P with $|P|$ minimum. Then P is irreducible, therefore indecomposable. By Lemma 2.2.2, $\text{width}(P) \geq \dim(P) = 3$. Let A be a maximum antichain in P. Then we have $|A| \geq 3$. Theorem 2.2.4 implies $|P - A| \geq \dim(P) = 3$.

Case 1. $|A| \geq 5$.

Let $a_1, a_2, a_3, a_4, a_5 \in A$. Since $\{a_1, a_2\}, \{a_3, a_4\}$ is an antichain matching of size 2, we know that $x < a_5 < y$ for any $x \in D(A)$ and any $y \in U(A)$. By symmetry, we have $x < a_i < y$ for any $x \in D(A)$, $y \in U(A)$ and $a_i \in A$. It follows that A is an autonomous subposet of P, which contradicts the fact that P is indecomposable.

Case 2. $|A| = 4$.

Let $A = \{a_1, a_2, a_3, a_4\}$. Since $\{a_1, a_2\}, \{a_3, a_4\}$ is an antichain matching of size 2, we know that $P - A$ is a chain. If $U(A) = \emptyset$, then by Theorem 2.5.1, $\dim(P) \leq 1 + \text{width}(P - A) = 2$. Hence $U(A) \neq \emptyset$. Similarly, $D(A) \neq \emptyset$. Let $y \in U(A)$ be a maximal element of P. Since P is irreducible, y cannot be the only maximal element of P. Without loss of generality, let a_1 be another maximal element of P. It follows that $y \parallel a_1$. Similarly, let $x \in D(A)$ be a minimal element of P, and x cannot be the only minimal element of P. Note that if a_1 is also a minimal element of P, then a_1 must be a loose point, which is not possible. Hence we may assume, without loss of generality, that a_2 is another minimal element of P. It follows that $x \parallel a_2$. However, there is an antichain matching of size 3 in P, namely $\{a_1, y\}$, $\{a_2, x\}$ and $\{a_3, a_4\}$.

Case 3. $|A| = 3$.

Let $A = \{a_1, a_2, a_3\}$. Suppose $D(A) = \emptyset$, then $|P| \geq 6$ implies $|U(A)| \geq 3$. Since A is not an autonomous subposet of P, there exist $w \in U(A)$ and $a_i \in A$ such that $w \parallel a_i$. Without loss of generality, we may assume $w \parallel a_1$. Since $\{a_1, w\}, \{a_2, a_3\}$ is an antichain matching of size 2, we know that $P - (A \cup \{w\})$ is a chain. Note that $U(A)$ is covered by two chains, one of which is a single point chain $\{w\}$. By Theorem 2.5.4, $\dim(P) \leq \text{width}(U(A)) = 2$. This contradiction implies $D(A) \neq \emptyset$. For the same reason, $U(A) \neq \emptyset$. Since $|U(A) \cup D(A)| = |P - A| \geq 3$, we may assume, without loss of generality, that $|U(A)| \geq 2$ and $|D(A)| \geq 1$.

Claim 1. *$P - A$ is a chain.*

Proof. As P is irreducible and indecomposable, there exists $z \in D(A)$ and $a_i \in A$ such that $z \parallel a_i$. Without loss of generality, we may assume $z \parallel a_1$. Since $\{a_1, z\}, \{a_2, a_3\}$ is an antichain matching of size 2, we know that $P - (A \cup \{z\})$ is a chain. In particular, $U(A)$ is a chain. Dually, we can conclude that $D(A)$ is also a chain.

Let u be the maximal element of $U(A)$, and y be the minimal element of $U(A)$. Since $|U(A)| \geq 2$, we know that both u and y exists and $u \neq y$. Let x be the maximal element of $D(A)$. Since $|D(A)| \geq 1$, such x does exist. It then follows that $x \geq z$ and $x \parallel a_1$. As u cannot be the only maximal element of P, there exists $a_i \in A$ such that $u \parallel a_i$. If $y \parallel x$, then $\{y, x\}, \{u, a_i\}, \{a_{i+1}, a_{i+2}\}$ is an antichain matching of size 3. So we must have $y > x$. Therefore $P - A$ is a chain.

As a_1 is not a loose point, we have $u > a_1$. Therefore, we may assume, without loss of generality, that $u \parallel a_2$. Since $\{a_1, x\}, \{a_2, u\}$ is an antichain matching of size 2, we know that $a_3 < y$. Note that if $x < a_3$, then $p - \{a_1, a_2\}$ is a chain. We can attach a "zero" and a "one" to P so that the resulting poset is planar. This is not possible as Theorem 2.6.3 implies $\dim(P) \leq 2$. Hence we must have $x \parallel a_3$.

Now $x \parallel a_1, a_3$ implies $x < a_2$. Since $\{a_3, x\}, \{a_2, u\}$ is an antichain matching

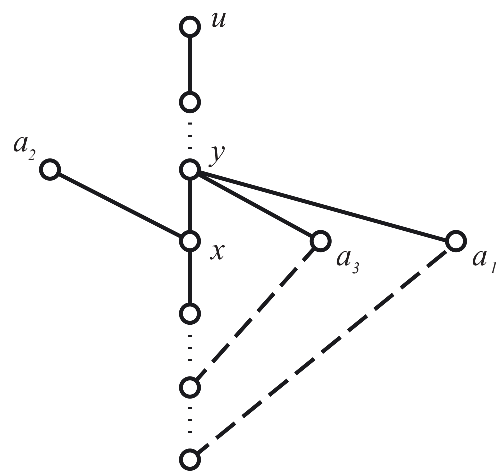

Figure 11: The dimension of P is at most 2

of size 2, we know that $a_1 < y$. Again, We can attach a "zero" and a "one" to P (Figure 11) so that the resulting poset is planar. Then we have $\dim(P) \leq 2$ followed by Theorem 2.6.3. The contradiction completes the proof of the Theorem 2.7.1. $\square$

CHAPTER III

GRAPHS WITH LARGE GIRTH AND LARGE CHROMATIC NUMBER

3.1 Introduction

The existence of graphs with large girth and large chromatic number is a classic combinatorial problem. For small girth, Tutte [7] and Zykov [47] first proved that there exist triangle free graphs with large chromatic number.

Let G be a graph with some large girth g. Then starting from any vertex of G, locally G looks like a tree. As a tree is 2-colorable, intuitively one might think that G can be colored with small number of colors. However, this is far from being true. In 1959, Erdős [18], in a landmark paper, proved the existence of graphs with arbitrarily large girth and arbitrarily large chromatic number using probabilistic method.

Theorem 3.1.1. *For all positive integers k and g, there exists a graph G with* $\mathrm{girth}(G) > g$ *and* $\chi(G) > k$.

The classic and elegant proof uses Erdős-Rényi graph $G \sim G(n,p)$, a random graph on n vertices and each edge is present in G with probability p independent from every other edge. The probability p is carefully selected so that once n is sufficiently large, with large probability (greater than 1/2), the number of cycles of size at most g is small (say less than $n/2$), and the size of the largest independent set in G is small. Then with positive probability there exists a graph G with less than $n/2$ cycles of length at most g and with small independence number. Remove from G a vertex from each cycle of length at most g, the resulting graph satisfies Theorem 3.1.1.

Although the existence of graphs with large girth and large chromatic number was proved by Erdős in 1959 using probability method, the first construction of such graphs was found by Lovász [33] in 1968. In 1979, Nešetřil and Rödl [35] gave a simple construction of graphs with large girth and large chromatic number. Both of their constructions are based on hypergraphs.

3.1.1 Cover Graphs

Recall that given a poset P, we can then associate with P a cover graph G_P on the same ground set as P, so that xy is an edge in G_P if and only if x covers y in P or y covers x in P. Note that cover graphs are triangle-free. Some thirty years ago, I. Rival asked whether there are cover graphs with large chromatic number. Bollobás noted in [12] that B. Descartes' classic proof [6] of the existence of triangle-free graphs with large chromatic number provided a positive answer. In fact, this construction shows that for each $r \geq 1$, there is a poset P of height r so that the chromatic number of the cover graph of P is r. In view of the dual form of Dilworth's theorem, this is the minimum value of height for which such a poset can possibly exist.

In [12], Bollobás showed that there are lattices whose cover graphs have arbitrarily large chromatic number. The example he constructed is complex and have a large dimension. Nešetřil and Trotter then asked the following question.

Question 3.1.2. *For every $k \geq 1$, is there a $r(k)$ such that for any poset P whose cover graph has chromatic number at least r, we have $\dim(P) \geq k$?*

Kříž and Nešetřil [32] answered Question 3.1.2 negatively for the case $k \geq 3$. They proved the following theorem.

Theorem 3.1.3. *For every $r \geq 1$, there exists a poset P with $\dim(P) \leq 2$ so that the chromatic number of the cover graph of P is r.*

3.1.2 A New Graph Parameter

On the way of proving Theorem 3.1.3, Kříž and Nešetřil introduced a new parameter for graphs, which we call the *eye parameter*. Formally, the eye parameter of a graph G, denoted $\text{eye}(G)$, is the least positive integer s for which there exists a family $\{L_1, L_2, \ldots, L_s\}$ of linear orders on the vertex set of G for which if x, y and z are three distinct vertices of G with $\{y, z\}$ an edge of G, then there is some i with $1 \leq i \leq s$ for which x is not between y and z in L_i. In this definition, it is allowed that x be *above* both y and z or *below* both y and z. For example, when G is a path, $\text{eye}(G) = 1$.

There is a notion of *dimension* of a graph G. The dimension of a graph G, denoted $\dim(G)$, is the least positive integer t for which there is a family $\{L_1, L_2, \ldots, L_t\}$ of linear orders on the vertex set of G so that the following two conditions are satisfied:

1. If x, y and z are distinct vertices of G and $\{y, z\}$ is an edge in G, then there is some i with $1 \leq i \leq t$ for which both $x > y$ and $x > z$ in L_i.

2. If x and y are distinct vertices of G, then there is some i with $1 \leq i \leq t$ so that $x > y$ in L_i.

Clearly we have $\text{eye}(G) \leq \dim(G)$. On the other hand, if a family of linear orders $\{L_1, L_2, \ldots, L_s\}$ witnesses $\text{eye}(G) = s$, then $\{L_1, L_2, \ldots, L_s, L'_1, L'_2, \ldots, L'_s\}$ is a family of linear orders that implies $\dim(G) \leq 2s$.

Proposition 3.1.4. *Let G be a graph. Then* $\text{eye}(G) \leq \dim(G) \leq 2\,\text{eye}(G)$.

If G is the cover graph of poset P with $\dim(P) \leq 2$, then $\text{eye}(G) \leq 2$, as the family of linear extensions which witnesses $\dim(P) \leq 2$ also witnesses $\text{eye}(G) \leq 2$. So as Kříž and Nešetřil noted, we have the following immediate corollary of Theorem 3.1.3.

Corollary 3.1.5. *For every $r \geq 1$, there is a graph G with* $\text{eye}(G) \leq 2$ *and* $\chi(G) = r$.

The graphs constructed by Kříž and Nešetřil in the proof of Theorem 3.1.3 and Corollary 3.1.5 have girth four. However, they were able to prove the following extension.

Theorem 3.1.6. *For every pair (g, r) of positive integers, there is a graph G with* $\text{girth}(G) \geq g$*,* $\chi(G) = r$ *and* $\text{eye}(G) \leq 3$*.*

In the same paper, Kříž and Nešetřil asked whether the last condition can be strengthened to $\text{eye}(G) \leq 2$.

Question 3.1.7. *For every pair (g, r) of positive integers, does there exist a graph G with* $\text{girth}(G) \geq g$*,* $\chi(G) = r$ *and* $\text{eye}(G) \leq 2$*?*

Question 3.1.7 is related to the famous Erdős-Hajnal conjecture [18]: There exists a smallest integer $f(r, g)$ for which every graph of chromatic number at least $f(r, g)$ contains a subgraph of girth at least g and chromatic number at least r.

Suppose the Erdős-Hajnal conjecture is true and such integer $f(r, g)$ does exist. Corollary 3.1.5 implies that for every pair (g, r) of positive integers, there is a graph G with $\text{eye}(G) \leq 2$ and $\chi(G) = f(r, g)$. Note that if H is a subgraph of G, then $\text{eye}(H) \leq 2$. So by the Erdős-Hajnal conjecture, there is a subgraph H of G such that $\text{eye}(H) \leq 2$, $\chi(H) \geq r$ and $\text{girth}(H) \geq g$. So the answer to Question 3.1.7 must be "Yes". On the other hand, if the answer to Question 3.1.7 is "No", then the Erdős-Hajnal conjecture fails.

3.2 Main Theorem

We answer Question 3.1.7 of Kříž and Nešetřil in the affirmative, which of course leaves the Erdős-Hajnal conjecture intact. The theorem we have is actually slightly stronger.

Theorem 3.2.1. *For every pair (g, r) of positive integers, there is a poset $P = P(g, r)$ with cover graph $G = G(g, r)$ so that the height of P is r, while* $\text{girth}(G) \geq g$ *and*

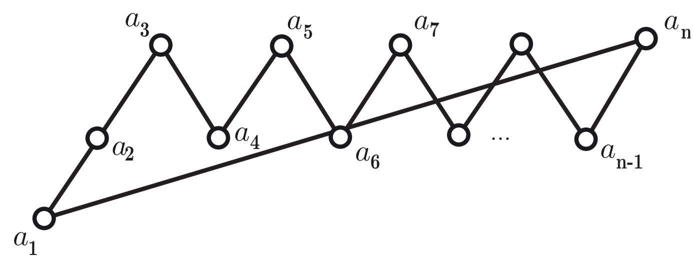

Figure 12: $P = P(g, 3)$ has eye parameter at most 2

$\chi(G) = r$. *Furthermore, there are two linear extensions* L_1 *and* L_2 *of* P *witnessing that* $\mathrm{eye}(G) \leq 2$.

Note that we do not claim that the poset $P(g, r)$ in Theorem 3.2.1 is 2-dimensional. In fact, the dimension of $P(g, r)$ grows rapidly with r, even with g fixed. We will return to this issue in the last section of this chapter.

3.2.1 Proof of the Main Theorem

We fix an integer $g \geq 4$ and then argue by induction on r. The basic idea behind the proof will be to make a minor adjustment to the construction used by Nešetřil and Rödl in [35].

The cases $r = 1$ and $r = 2$ are trivial. For $r = 1$, we let P be a single point. For $r = 2$, we let P be a two-point chain. To handle the case $r = 3$, we let n be an odd integer with $n \geq g$. Then we take $G = G(g, 3)$ as an odd cycle with vertex set $\{a_1, a_2, \ldots, a_n\}$, with $\{a_i, a_{i+1}\}$ an edge for each $i = 1, 2, \ldots, n-1$. Also, $\{a_n, a_1\}$ is an edge of G. Then we take $P = P(g, 3)$ as a poset whose cover graph is G by setting the following covering relations in P (Figure 12):

$$a_1 < a_2 < a_3 > a_4 < a_5 > a_6 < a_7 > a_8 < a_9 > \cdots > a_{n-1} < a_n > a_1.$$

We then take

$$L_1 = a_1 < a_2 < a_4 < a_3 < a_6 < a_5 < a_8 < a_7 < \cdots < a_{n-1} < a_{n-2} < a_n$$

and

$$L_2 = a_{n-1} < a_1 < a_n < a_{n-3} < a_{n-2} < \cdots < a_6 < a_7 < a_4 < a_5 < a_2 < a_3.$$

It is easy to see that L_1 and L_2 are linear extensions of P. Furthermore, the two endpoints of an edge in G occur consecutively in either L_1 or L_2, except for the edge $\{a_{n-1}, a_n\}$. However, only a_{n-2} is between a_{n-1} and a_n in L_1. Also, only a_1 is between a_{n-1} and a_n in L_2. It follows that L_1 and L_2 witness that $\mathrm{eye}(G) \leq 2$. Note that $\dim(P) \neq 2$. In fact, $\dim(P) \geq 3$.

Now suppose that for some $r \geq 3$, we have constructed a poset $P = P(g, r)$ with cover graph $G = G(g, r)$ so that the height of P is r, while $\mathrm{girth}(G) \geq g$ and $\chi(G) = r$. Suppose further that L_1 and L_2 are linear extensions of P witnessing that $\mathrm{eye}(G) \leq 2$.

We now explain how to construct a poset $Q = Q(g, r+1)$ with cover graph $H = H(g, r+1)$ so that the height of Q is $r+1$, while $\mathrm{girth}(H) \geq g$ and $\chi(H) = r+1$. We will also construct linear extensions M_1 and M_2 of Q witnessing that $\mathrm{eye}(H) \leq 2$.

Let A denote the vertex set of G and let $n = |A|$. Using the results of Nešetřil and Rödl as developed in [35], we know there exists a hypergraph $\mathcal{H}$ satisfying the following conditions: $\mathcal{H}$ is a simple n-uniform hypergraph; the girth of $\mathcal{H}$ is at least g; and the chromatic number of $\mathcal{H}$ is $r+1$. Let B and $\mathcal{E}$ denote, respectively, the vertex set and the edge set of $\mathcal{H}$. In the discussion to follow, we consider each edge $E \in \mathcal{E}$ as an n-element subset of B.

The poset Q is assembled as follows. Set $Z = \mathcal{E} \times A$. The ground set of Q will be $B \cup Z$ with all elements of B maximal in Q. For each edge E in $\mathcal{E}$, the elements of $\{E\} \times A$ determine a subposet of Q which we will denote $P(E)$. When a and a' are distinct elements of A, we will set $(E, a) < (E, a')$ in Q if and only if $a < a'$ in P. Accordingly, for each $E \in \mathcal{E}$, the subposet $P(E)$ is isomorphic to P. Also, when $E, E' \in \mathcal{E}$ and $E \neq E'$, we make all elements of $P(E)$ incomparable with all elements of $P(E')$.

We pause to point out that regardless of how the comparabilities between B and Z are defined in Q, for each edge $E \in \mathcal{E}$, the covering edges of $P(E)$ are covering edges in Q and these edges form a copy of G.

We now describe these comparabilities between B and Z. This will be done by prescribing when an element $b \in B$ covers an element $(E, a) \in Z$. We begin by choosing an arbitrary linear order $L(B)$ on B. Also, let $\{a_1, a_2, \ldots, a_n\}$ be a labelling of A so that L_1 is the subscript order, i.e., $a_i < a_j$ in L_1 if and only if $i < j$. Next, we fix an edge $E \in \mathcal{E}$ and describe the cover relations between B and $P(E)$. This process will be repeated for each edge $E \in \mathcal{E}$ and when this step has been completed, the poset Q is fully determined. First, when $b \in B - E$, we make b incomparable to all elements of $P(E)$ in Q. Second, let $\{b_1, b_2, \ldots, b_n\}$ be the labelling of the elements of E so that $b_i < b_j$ in $L(B)$ if and only if $i < j$. Then for each $i = 1, 2, \ldots, n$, we make b_i cover (E, a_i) in Q. It follows that if $(E, a) \in Z$, then there is a unique element $b \in B$ so that b covers (E, a) in Q.

Now that $Q = Q(g, r+1)$ has been defined, we take $H = H(g, r+1)$ as the cover graph of Q, and we need to show that the height of Q is $r+1$, while $\mathrm{girth}(H) \geq g$ and $\chi(H) = r+1$. Furthermore, there are two linear extensions M_1 and M_2 of Q witnessing that $\mathrm{eye}(H) \leq 2$.

Claim 1. $\chi(H) = r + 1$.

Proof. We note that $\chi(H) \geq r$, since H contains copies of G. On the other hand, it is trivial that we may color all elements of Z with r colors and use one new color on the independent set B, so that $\chi(H) \leq r + 1$. Now suppose that $\chi(H) = r$, and let ϕ be a proper coloring of H using r colors. Then since the chromatic number of $\mathcal{H}$ is $r+1$, there is some edge E of $\mathcal{H}$ on which ϕ is constant. This implies that ϕ colors the cover graph of $P(E)$ with only $r - 1$ colors, which is impossible. The contradiction shows that $\chi(H) = r + 1$, as desired.

Claim 2. $\mathrm{height}(Q) = r + 1$.

Proof. First, we note that the height of Q is at most $r+1$, since we have added B as a set of maximal elements to a family of pairwise disjoint and incomparable copies of P. On the other hand, we have shown that $\chi(H) = r+1$. So the height of H must be $r+1$, using the dual form of Dilworth's theorem.

Claim 3. $\text{girth}(H) \geq g$.

Proof. Consider a cycle C in H. If there is an edge E of $\mathcal{H}$ so that C is contained entirely within the cover graph of $P(E)$, then it has size at least g. So we may assume that C involves vertices from copies of P associated with two or more edges in $\mathcal{E}$. Now the fact that the covering edges between B and each $P(E)$ are formed using a bijection means that once the cycle enters some $P(E)$, it must pass through at least two vertices before leaving. So the girth requirement is satisfied (generously) be the fact that the girth of $\mathcal{H}$ is at least g. In this detail, we point out that we are using essentially the same idea as in [35].

Claim 4. *There are two linear extensions M_1 and M_2 of Q witnessing that* $\text{eye}(H) \leq 2$.

Proof. In fact, there is considerable flexibility in how this is done. For each $b \in B$, let $N(b)$ denote the set of all elements (E, a) from Z such that b covers (E, a) in Q, i.e., $N(b)$ is just the neighborhood of b in the cover graph H. Note that $N(b)$ is an antichain in the poset Q.

Let $L(\mathcal{E})$ be an arbitrary linear order on $\mathcal{E}$. We define linear extensions M_1 and M_2 by the following rules (starting with the rules for M_2):

1. The restriction of M_2 to B is an arbitrary linear order. In M_2 all elements of Z are below all elements of B. Furthermore, if (E, a) and (E', a') are distinct elements of Z, then $(E, a) < (E', a')$ in M_2 if and only if either $E < E'$ in $L(\mathcal{E})$ or $E = E'$ and $a < a'$ in L_2.

2. The restriction of M_1 to B is the linear order $L(B)$. In M_1, for each $b \in B$, all elements of $N(b)$ will be placed in the gap immediately under b and above all other elements (if any) of B which are under b in $L(B)$. The restriction of M_1 to $N(b)$ will be the dual of the restriction of M_2 to $N(b)$.

We need to show that M_1 and M_2 are linear extensions of Q, and we remark that it is enough to show that they both respect the covering relations in Q. First, we note that for each $i = 1, 2$, and for each $E \in \mathcal{E}$, if a and a' are distinct elements of A, then $(E, a) < (E, a')$ in M_i if and only if $a < a'$ in L_i. On the other hand, if $b \in B$, $(E, a) \in Z$ and b covers (E, a) in Q, then $(E, a) \in N(b)$ so it is placed below b in M_1. Finally, we note that all elements of Z are below all elements of B in M_2. We conclude that M_1 and M_2 are linear extensions of Q, as desired.

Finally, we explain why M_1 and M_2 witness that $\mathrm{eye}(H) \leq 2$. Consider how an edge might possibly trap a vertex in both M_1 and M_2. If the edge is an edge in the cover graph of some $P(E)$, then the linear extension M_2 forces the vertex to also belong to $P(E)$. But the restriction of M_1 and M_2 to $P(E)$ are just like L_1 and L_2 for G, so this situation cannot lead to a problem.

Similarly, if the edge joins some $b \in B$ to a vertex (E, a) in $N(b)$, then the only potential problem is a vertex $(E', a') \in N(b)$ with $(E, a) < (E', a') < b$ in M_1. However, the rules for M_1 and M_2 imply that $(E', a') < (E, a) < b$ in M_2. This completes the proof of Claim 4 and Theorem 3.2.1.

3.3 Upper and Lower Cover Dimension

The following lemma is straightforward.

Lemma 3.3.1. *Let P be a poset and let G be the cover graph of P. If $\dim(P) = t$, then $\dim(G) \leq 2t$.*

Proof. Let $t = \dim(P)$ and let $\mathcal{R} = \{L_i : 1 \leq i \leq t\}$ be a realizer of P. Then for each

$i = 1, 2, \ldots, t$, let L_i^* be the dual of L_i, i.e., $x > y$ in L_i^* if and only if $x < y$ in L_i. We claim that the family $\mathcal{R}^* = \mathcal{R} \cup \{L_i^* : 1 \leq i \leq t\}$ witnesses that $\dim(G) \leq 2t$.

To see this, note that the second condition of dimension of a graph holds since L_1 and L_1^* are in the family. Now let x, y and z be distinct vertices with $\{y, z\}$ an edge in G. Without loss of generality, we take $y < z$ in P. If $x \not< z$ in P, then there is some i with $1 \leq i \leq t$ so that $x > z$ in L_i. This implies $x > z > y$ in L_i. So we may assume that $x < z$ in P. Since $\{y, z\}$ is an edge of the cover graph, we cannot have $y < x$ in P. It follows that there is some j with $1 \leq j \leq t$ so that $x < y$ in L_j. This implies that $x < y < z$ in L_j and $x > y > z$ in L_j^*. This completes the proof of the lemma. □

We believe the bound in the above basic lemma is tight.

Conjecture 3.3.2. *For every $t \geq 1$, there exists a poset P of dimension t so that if G is the cover graph of P, then* $\dim(G) = 2\dim(P)$.

It can be shown that Conjecture 3.3.2 holds when $t \leq 2$. For the case $t = 1$, let P be a chain of height 2 and let G be the cover graph of P. Then $\dim(P) = 1$ and $\dim(G) = 2$.

For the case $t = 2$, by Theorem 3.1.3, there exists a poset P with cover graph G so that $\dim(P) = 2$ and $\chi(G) = 5$. As G is not 4-colorable, four color theorem implies that G is not a planar graph. Then the following Schnyder's theorem [37] implies $\dim(G) \geq 4$. Therefore by Lemma 3.3.1, $\dim(G) = 4$.

Theorem 3.3.3. *Let G be a graph. Then G is planar if and only if* $\dim(G) \leq 3$.

Conjecture 3.3.2 is still open for larger values of t. However, our preliminary thoughts on this conjecture suggest a more extensive line of research.

For a poset P, we call a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P an *upper-cover realizer of P* provided that whenever (z, x, y) is an ordered triple of distinct points in P with z covering both x and y, there is some i with $x > y$ in L_i. (Note that x and y are incomparable in P, as z covers both x and y in P.) The *upper cover dimension* of P, denoted $\dim_{uc}(P)$, would then be the minimum size of an upper-cover realizer of P.

Lower-cover realizer and the *lower cover dimension* of P, denoted $\dim_{lc}(P)$, would then be defined dually. Clearly, $\dim_{uc}(P) \leq \dim(P)$ and $\dim_{lc}(P) \leq \dim(P)$.

An attractive feature of these new parameters is that they are monotonic on subdiagrams of the order diagram of P, i.e., if we consider the diagram D of P as an acyclic orientation of the cover graph G, and D' is a subdiagram of D, then D' determines a poset P' which is a suborder of P. On the one hand, P' is not necessarily a subposet of P and it is clear that $\dim(P')$ may be the same as $\dim(P)$ or arbitrarily smaller or larger. On the other hand, if edges are removed from the diagram, then upper and lower cover dimension cannot increase, i.e., $\dim_{uc}(P') \leq \dim_{uc}(P)$ and $\dim_{lc}(P') \leq \dim_{lc}(P)$.

Although the difference between upper (lower) dimension and dimension of a poset can be arbitrarily large, for example, the standard example $S_n (n \geq 3)$ has dimension n and both of its upper and lower dimensions are 2. There are infinitely many posets for which the upper dimension, the lower dimension and the dimension are the same.

Lemma 3.3.4. *Let P be a poset on n points and $\dim(P) = d \geq 3$. Then there exists a poset Q on at most n^2 points so that Q contains P as a subposet, and*

$$\dim_{uc}(Q) = \dim_{lc}(Q) = \dim(Q) = \dim(P).$$

Proof. The proof idea is straightforward. Starting from original poset P, each time we add one more point to current poset, until the upper and lower dimension of the new poset match the dimension of P. At the same time, the dimension of the new poset, which initially equals to the dimension of P, will stay unchanged.

Let X be the ground set of P. Let $\mathcal{F} = \{L_1, L_2, \ldots, L_d\}$ be a realizer of P.

Suppose $\dim_{uc}(P) < \dim(P)$. Then there exists an pair $(x, y) \in \mathrm{Inc}(P)$ so that the upper covers of x and y are disjoint in P. We add b_{xy} to P, and let $P' = P \cup \{b_{xy}\}$.

Instead of specifying where to add b_{xy} to P, we will build a realizer of P' based on $\mathcal{F}$. Then P' can be recovered from its realizer. As x and y are incomparable in P, there exists two linear extension L_i and L_j in $\mathcal{F}$ such that $x < y$ in L_i and $y < x$ in L_j. Let $\mathcal{F}' = \{L'_1, L'_2, \ldots, L'_d\}$ be a realizer of P'. In L'_i, b_{xy} is inserted right above y, and the restriction of L'_i to X is L_i. Similarly, In L'_j, b_{xy} is inserted right above x, and the restriction of L'_j to X is L_j. In linear extension L'_k where $k \neq i, j$, b_{xy} is inserted at the top, and the restriction of L'_k to X is L_k.

We pause to mention that as the restriction of $\mathcal{F}'$ to X is $\mathcal{F}$, P is a subposet of P'. Hence $d = \dim(P) \leq \dim(P')$. On the other hand, P' has a realizer of size d as witnessed by $\mathcal{F}'$, so we conclude $\dim(P') = \dim(P) = d$.

We note that as $d \geq 3$, there exist $k \neq i, j$ such that b_{xy} is at the top of L'_k, so b_{xy} is a maximal element in P'. Therefore, every cover relation in P is preserved in P'. Hence we have $\dim_{uc}(P') \geq \dim_{uc}(P)$ and $\dim_{lc}(P') \geq \dim_{lc}(P)$. Moreover, in P', b_{xy} covers both x and y (and possibly some other points). Suppose not, then there exists $z \in P$ so that, without loss of generality, $b_{xy} > z > x$ in P'. However, this cannot be true as by our construction of $\mathcal{F}'$, we have $z > b_{xy} > x$ in L'_j.

Now we repeat the above process of adding one point at a time as long as there exists $u, v \in P$ such that u and v are incomparable in P, and their upper covers are disjoint in the current poset. Note that in each iteration, 1) the new poset contains the old one as a subposet; 2) the dimension of the new poset will stay unchanged, 3) the upper and lower dimension of the new poset will not decrease; 4) the new poset preserves all the cover relations in the old poset.

Let P^* be the resulting poset. Note that P is a subposet of P^*. Also, we have $\dim(P^*) = \dim(P) = d$. Furthermore, for every incomparable pair $(u, v) \in \mathrm{Inc}(P)$, there exists $z \in P^*$ so that z covers both x and y in P^*. Therefore, $\dim_{uc}(P^*) =$

$\dim(P) = d$. Since there are at most $n(n-1)/2$ ordered incomparable pairs in P, we conclude that $|P^*| \leq n + n(n-1)/2$.

The lower cover dimension will be exactly the dual. Each time, instead of adding a maximal element b_{xy}, we add a minimal element a_{xy}. After adding at most $n(n-1)/2$ points to P^*, the final poset Q will have at most $n + n(n-1)/2 + n(n-1)/2 = n^2$ points. It also satisfies $\dim_{uc}(Q) = \dim_{lc}(Q) = \dim(Q) = \dim(P) = d$. $\square$

We make the following conjecture, which is stronger than Conjecture 3.3.2.

Conjecture 3.3.5. *For every pair (d, r) of positive integers, there is a poset P with $\dim(P) = d$ so that if D is the order diagram of P, E is the edge set of D and $\phi : E \to \{1, 2, \ldots, r\}$ is an r coloring of the edges of D, then there is some $\alpha \in \{1, 2, \ldots, r\}$ so that if we take D' as the subdiagram of D with edge set $\{e \in E : \phi(e) = \alpha\}$ and set P' as the suborder of P determined by D', then $\dim_{uc}(P') = \dim_{lc}(P') = \dim(P)$.*

Lemma 3.3.6. *Conjecture 3.3.5 implies Conjecture 3.3.2. i.e., if Conjecture 3.3.5 is true, so is Conjecture 3.3.2.*

Proof. Suppose Conjecture 3.3.5 is true. For any $t \geq 1$, let $r = 2^{2t-1}$ and let P be the poset satisfies Conjecture 3.3.5. Let G be the cover graph of P. We will show that $\dim(G) = 2\dim(P) = 2t$. Suppose not, by Lemma 3.3.1, we have $\dim(G) \leq 2\dim(P) - 1 = 2t - 1$. Let $\mathcal{F} = \{L_1, L_2, \ldots, L_{2t-1}\}$ be a family of linear orders which witnesses $\dim(G) \leq 2t - 1$.

Since D is the order diagram of P, each edge of D is a cover edge of P. For each edge $e = \{x < y\}$ of D, color e by $r = 2^{2t-1}$ colors by assigning to e a bit string of length $2t - 1$. Coordinate i of this bit string is 0 if $x < y$ in L_i , and 1 if $x > y$ in L_i. By Conjecture 3.3.5, there is some bit string α so that if P' is the poset determined by the edges with color α, then $\dim_{uc}(P') = \dim_{lc}(P') = \dim(P) = t$.

Let X denote the ground set of P'. Let $\mathcal{F}' = \{L'_1, L'_2, \ldots, L'_{2t-1}\}$ be a family of linear orders where L'_i is the restriction of L_i on X. Note that as every cover edge of P' has color α, when coordinate i of the string α is 0, the linear order L'_i is a linear extension of P'. Similarly, when coordinate i of the string α is 1, the dual of the linear order L'_i is a linear extension of P'. Let $\mathcal{F}_0 = \{L'_i \in \mathcal{F}' : \alpha(i) = 0\}$ and $\mathcal{F}_1 = \{L'_i \in \mathcal{F}' : \alpha(i) = 1\}$. We have $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$ and $\mathcal{F}_0 \cup \mathcal{F}_1 = \mathcal{F}'$.

We claim $\mathcal{F}_0$ is a lower cover realizer of P'. For any lower cover configuration $\{z, x, y\}$ in P' where z is covered by both x and y, as $\mathcal{F}$ witnesses $\dim(G) \leq 2t-1$, there exist L_i and L_j in $\mathcal{F}$ such that $x > y, z$ in L_i and $y > x, z$ in L_j. Therefore we have $x > y, z$ in L'_i and $y > x, z$ in L'_j. However, every linear order in $\mathcal{F}'$ is either a linear extension of P', or the dual of a linear extension of P'. Hence we must have $x > y > z$ in L'_i and $y > x > z$ in L'_j. Note that both L'_i and L'_j are linear extensions of P', therefore $L'_i, L'_j \in \mathcal{F}_0$. We conclude that $\mathcal{F}_0$ is a lower cover realizer of P'. Dually, $\mathcal{F}_1$ is a upper cover realizer of P'.

Since $\dim_{uc}(P') = \dim_{lc}(P') = \dim(P) = t$, we have $2t - 1 = |\mathcal{F}'| = |\mathcal{F}_0| + |\mathcal{F}_1| \geq t + t = 2t$. This contradiction completes the proof of Lemma 3.3.6.

$\square$

CHAPTER IV

DIMENSION OF RANDOM ORDERED SETS

4.1 Introduction

In 1991, Erdős, Kierstead and Trotter [19] investigated the dimension of random bipartite posets in the probability space $\Omega(n, p)$ defined as follows. Posets in $\Omega(n, p)$ have the form $P = A \cup A'$ and are called *bipartite posets*. Elements of A are minimal in P while elements of A' are maximal. Furthermore, for each pair $(a, a') \in A \times A'$, let $(a < a')$ be the event that $a < a'$ in P. Set $\Pr(a < a') = p$ (in general p is a function of n) with events corresponding to distinct pairs of $A \times A'$ independent.

Throughout this chapter, we use the acronym EKT to refer to the paper [19] by Erdős, Kierstead and Trotter. Also, our notation and terminology will be essentially the same as in EKT—with just a few minor variations.

We use the following standard terminology: When we say that a statement (an inequality, for example) involving an integer n holds *almost surely*, we mean that for every $\epsilon > 0$, there is an integer n_0 so that if $n > n_0$, the probability that the statement fails to hold is less than ϵ. Also, we will at times include the symbol ϵ as part of the statement. For example, $(1 - 1/n)^n < 1/e$ for all $n \geq 1$ while $(1 - 1/n)^n > (1 - \epsilon)/e$, almost surely.

With this discussion in mind, here are the statements of the principal results in EKT, starting with a comprehensive lower bound.

Theorem 4.1.1. *For every $\epsilon > 0$, there is a constant $\delta > 0$ so that if*

$$\frac{\log^{1+\epsilon} n}{n} < p < 1 - n^{-1+\epsilon},$$

then almost surely,

$$\dim(P) > \frac{\delta pn \log pn}{1 + \delta p \log pn}.$$

The proof of Theorem 4.1.1 is obtained via a single, and quite complex, argument. However, it may be more natural to split the statement of its consequences into two separate ranges.

Corollary 4.1.2. *For every $\epsilon > 0$, there is a constant $\delta > 0$ so that if*

$$\frac{\log^{1+\epsilon} n}{n} < p \leq \frac{1}{\log n},$$

then almost surely,

$$\dim(P) > \delta pn \log pn.$$

Corollary 4.1.3. *For every $\epsilon > 0$, there is a constant $C > 0$ so that if*

$$\frac{1}{\log n} \leq p < 1 - n^{-1+\epsilon},$$

then almost surely,

$$\dim(P) > n - \frac{Cn}{p \log n}.$$

We pause to comment on the historical significance of the lower bound in Corollary 4.1.2. For a poset P (any poset, not just a height 2 poset), let

$$\Delta(P) = \max_{x \in P} |\{y \in P : y < x \text{ in } P \text{ or } y > x \text{ in } P\}|.$$

Just as is the case in graph theory, the quantity $\Delta(P)$ is called the *maximum degree* of P. Then we have the following upper bounds for dimension in terms of maximum degree proven by Füredi and Kahn [21].

Theorem 4.1.4. *There exists a positive constant C_1 so that if P is a poset on n points and $k = \Delta(P)$, then*

$$\dim(P) < C_1 k \log n.$$

Theorem 4.1.5. *There exists a positive constant C_2 so that if P is a poset on n points and $k = \Delta(P)$, then*

$$\dim(P) < C_2 k \log^2 k.$$

Theorem 4.1.4 is proved using relatively straightforward probabilistic techniques, while the proof of Theorem 4.1.5 requires the Lovász Local Lemma [20]. As noted in EKT, when $n^{-1+\epsilon} < p \le 1/\log n$, $\log(pn) = \Theta(\log n)$, so a poset P in $\Omega(n, p)$ satisfies, almost surely, the twin inequalities: $\Delta(P) \le 2pn$ and $\dim(P) > \delta pn \log n$. This shows that the inequality in Theorem 4.1.4 is best possible—up to a multiplicative constant.

In their seminal 1941 paper where the concept of dimension is introduced, Dushnik and Miller [16] gave the following construction: For each $n \ge 2$, $S_d = A \cup A'$ is the bipartite poset with $A = \{a_1, a_2, \dots, a_d\}$, $A' = \{a'_1, a'_2, \dots, a'_d\}$ and $a_i < a'_j$ in S_d if and only if $i \ne j$. Dushnik and Miller noted that $\dim(S_d) = d$, and S_d is now referred to as the *standard example* of a poset of dimension d. We will discuss standard examples more extensively in Sections 4.5 and 4.6.

For an integer $k \ge 1$, let m_k be the maximum dimension of a poset P with $\Delta(P) = k$. Trivially, $m_1 = 2$, and it is an easy exercise to show that $m_2 = 3$. However, the exact value of m_3 is not known. In general, the standard examples show $m_k \ge k + 1$, but until the early 1980's, it was not even known whether the parameter m_k is well defined. However, Rödl and Trotter found a simple proof that m_k exists and satisfies $m_k \le 2k^2 + 2$. The second of the two Füredi-Kahn inequalities, which was published in 1984, asserts a much stronger result: $m_k = O(k \log^2 k)$.

However, the lower bound in Corollary 4.1.2 also shows that $m_k = \Omega(k \log k)$. Determining the true behavior of the ratio m_k/k remains one of the most challenging (and probably quite difficult) problems in dimension theory. Also, in over 20 years, no one has yet provided an explicit construction for a poset P with $\Delta(P) = k$ and $\dim(P) \ge k + 2$, when $k \ge 3$.

As for upper bounds, in view of the twin Füredi-Kahn bounds, EKT is concerned only with the range $1/\log n \leq p < 1$ where they provide the following result.

Theorem 4.1.6. *For every $\epsilon > 0$, if $1/\log n \leq p < 1$, then almost surely*

$$\dim(P) < n - \frac{n \log(1/p)}{(2+\epsilon)\log n}.$$

Combining the two bounds, when p is a constant, it follows that there are positive constants c_1 and c_2 (depending on p) so that almost surely, $\dim(P)$ is in the range:

$$n - c_1 \frac{n}{\log n} < \dim(P) < n - c_2 \frac{n}{\log n}.$$

While essentially all of the analysis in EKT is done for bipartite posets, they do discuss the model where all labelled posets with ground set $\{1, 2, \ldots, n\}$ are considered as equally likely outcomes. For this model, EKT then provides positive constants c_1' and c_2' so that, almost surely,

$$\frac{n}{4} - c_1' \frac{n}{\log n} < \dim(P) < \frac{n}{4} - c_2' \frac{n}{\log n}.$$

4.1.1 Statements of New Results

As is clear from the preceding discussion, The research in EKT was motivated primarily by two goals. One was to analyze the accuracy of the Füredi-Kahn upper bounds on dimension in terms of maximum degree. The second was to develop the machinery for estimating the expected dimension of a random labelled poset on n points. For these reasons, most of the effort in EKT was focused on the case $0 < p \leq 1/2$. While bounds are given in EKT for the range $1/2 \leq p < 1$, where the expected value of dimension is $(1 - o(1))n$, the accuracy of their results deteriorates as p gets close to 1.

Our primary focus here will be on the range $1/2 \leq p < 1$, where it is often the case that statements are more elegant and the analysis is somewhat cleaner when we work with the complementary parameter $q = 1 - p$. So throughout this section, we use the symbol $q = 1 - p$ without further comment. The reader may note that some of our results extend to the case $p < 1/2$, but here the result reduces to (essentially) the same result already proved in EKT. So unless there is good reason not to do so, we have elected to add $p \geq 1/2$ (equivalently, $q \leq 1/2$) to the hypothesis of almost all of our results.

Although the result is essentially a restatement of the the upper bound in Theorem 4.1.6, we will provide an elementary proof of the following result.

Theorem 4.1.7. *For every* $\epsilon > 0$*, if*

$$\frac{\log^{1+\epsilon} n}{n} < q \leq 1/2,$$

then almost surely,

$$\dim(P) \leq n - \frac{qn}{(2+\epsilon)\log n}.$$

Furthermore, our proof of Theorem 4.1.7 provides the framework for the following more general upper bound, stated in terms of the Euler function:

$$\phi(q) = \prod_{i=1}^{\infty}(1 - q^i).$$

Theorem 4.1.8. *For every* $\epsilon > 0$*, if*

$$\frac{\log^{1+\epsilon} n}{n} < q \leq 1/2,$$

then almost surely,

$$\dim(P) < n - \frac{(\log(1/\phi(q)))n}{(2+\epsilon)\log n}.$$

Note that when $\log^{1+\epsilon} n/n < q \leq 1/2$, $\log(1/\phi(q)) > \log(1/p)$. So indeed Theorem 4.1.8 is a strengthening of Theorem 4.1.6. The gap between $\log(1/\phi(q))$ and $\log(1/p)$ is at 0.5489 when $q = 1/2$, and it tends to zero as q goes to zero.

We believe that the inequality in the preceding theorem is the "right answer", at least when q is not too small, and we make the following conjecture.

Conjecture 4.1.9. *For every $\epsilon > 0$, if*

$$n^{-1/3} \leq q \leq 1/2,$$

then almost surely,

$$\dim(P) > n - \frac{(\log(1/\phi(q))n}{(2-\epsilon)\log n}.$$

On the other hand, we also prove the following two lower bounds which are considerable strengthening of the results in EKT.

Theorem 4.1.10. *For every $\epsilon > 0$, if $C > 4/\epsilon$, and $n^{-1/4+\epsilon} \leq q \leq 1/2$, then almost surely,*

$$\dim(P) > n - \frac{Cqn}{\log n}.$$

Theorem 4.1.11. *For every $\epsilon > 0$, if $\log^{1+\epsilon} n/n < q \leq n^{-1/4}$, then almost surely,*

$$\dim(P) > n - \frac{n^{2/3}\log^{(\epsilon-2)/3} n}{q^{1/3}}.$$

Note that the lower bound in Theorem 4.1.10 is increasing as q tends to zero. Of course, this behavior is matched by the upper bound in Theorem 4.1.8. However, the lower bound in Theorem 4.1.11 is *decreasing* as q tends to zero.

Using techniques which are not represented in EKT, we will prove two additional bounds. The first of these results shows that the rebounding behavior of $\mathrm{Idim}(P)$ predicted by our lower bounds is correct, i.e., we can conclude that as q tends to zero, the expected value of dimension first increases and then decreases, a subtlety not identified in EKT.

Theorem 4.1.12. *For every $\epsilon > 0$, if*

$$\frac{\log^{1+\epsilon} n}{n} < q \leq n^{-1/2},$$

then almost surely,

$$\dim(P) \leq n - \frac{\log n}{q}.$$

The second result is an improvement in the lower bound when q is very small. It shows that the preceding bound is essentially best possible.

Theorem 4.1.13. *For every $\epsilon > 0$, if*

$$\frac{\log^{1+\epsilon} n}{n} < q \leq n^{-4/5},$$

then almost surely,

$$\dim(P) > n - \frac{4\log n}{q}.$$

As a consequence, the interval on q where we do not have tight control has been narrowed to $n^{-4/5} < q \leq n^{-1/4}$. Outside this range, we have upper and lower bounds on the expected value of $n - \dim(P)$ which differ by at most a multiplicative constant.

The remainder of this chapter is organized as follows. In the next section, we provide a concise summary of essential notation, terminology and background material. In Section 4.3, we provide a simple proof of Theorem 4.1.7 as well as the more complex argument for the new upper bound in Theorem 4.1.8. This section will also illustrate the connections with q-series, Euler functions and latin rectangles. In Section 4.4, we provide the proofs of our lower bounds in Theorems 4.1.10 and 4.1.11. These proofs will follow along lines which are close to the approach of EKT—with some minor changes and some simplification in the analysis.

In Section 4.5, we turn to a motivating extremal problem, and using new methods not present in EKT, we will prove the upper bound in Theorem 4.1.12 and the lower bound in Theorem 4.1.13. In particular, this upper bound shows that dimension does indeed "rebound" and shrink as q approaches 0. We also have a nice application of the asymmetric form of the Lovász Local Lemma when we apply our results to the extremal problem.

In Section 4.6, we consider a second extremal problem, where will actually be considering very small values of p. With the machinery built up, we can quickly make a major advance on this problem. We close in Section 4.7 with a brief discussion of some remaining open problems.

4.2 Essential Background Material

We follow the standard practice in combinatorial mathematics of saying, for example, that M is a set of size $n/\log n$, a statement that requires $n/\log n$ be an integer. The minor complexities associated with rounding up or rounding down as the situation requires can easily be handled—but in the interim, they only serve to obscure the line of reasoning.

In a bipartite poset $P = A \cup A'$, when $M \subseteq A$ and $M' \subseteq A'$, we let $\mathrm{Inc}(M, M')$ denote the set of all pairs $(a, a') \in M \times M'$ with $a \parallel a'$ in P. When L is a linear order on A, then for every $a \in A$, we let:

$$h_L(a) = |\{b \in A : b > a \text{ in } L\}|$$

denote the height of a in L, i.e., $h_L(a)$ is just the number of elements of A which are higher than a in L. Note that the top element of A in L is at height 0. When $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ is a family of linear orders on A and $a \in A$, we abbreviate $h_{L_j}(a)$ as $h_j(a)$.

When working with a linear order L' on A', the notation of height will be applied upside-down, i.e., $h'_{L'}(a')$ counts the number of elements of A' which are below a' in L'. So an element a' with $h'_{L'}(a') = 0$ is the lowest element of A' in L'.

4.2.1 Interval Dimension

For technical reasons which will soon be clear, our proofs will use two variants of dimension for a bipartite poset P—the conventional dimension $\dim(P)$ and the *interval dimension* $\mathrm{Idim}(P)$.

Here is the framework for the interval dimension. When $\mathcal{S} \subseteq A \times A'$, we say that a family $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P *reverses* $\mathcal{S}$ when the following property holds:

Reversal Property. For every pair $(a, a') \in \mathcal{S}$ with $a \parallel a'$ in P, there is some j with $1 \leq j \leq t$, so that $a > a'$ in L_j.

We then define the interval dimension $\mathrm{Idim}(P)$ as the least positive integer d for which there is a family $\mathcal{F} = \{L_1, L_2, \ldots, L_d\}$ of linear extensions of P which reverses $A \times A'$. Here the interval dimension was introduced by Trotter and Bogart in [11]. However, it is defined for all posets while our definition for $\mathrm{Idim}(P)$ only makes sense for bipartite posets.

Proposition 4.2.1. *For every bipartite poset $P = A \cup A'$,* $\mathrm{Idim}(P) \leq \dim(P) \leq \mathrm{Idim}(P) + 1$.

Proof. Clearly we have $\mathrm{Idim}(P) \leq \dim(P)$ by definition. Let $\mathcal{F} = \{L_1, L_2, \ldots, L_d\}$ be a family of linear extensions of P which reverses $A \times A'$, where $d = \mathrm{Idim}(P)$. Let $L_{d+1} = [L_1^*(A) < L_1^*(A')]$, and $\mathcal{F}' = \mathcal{F} \cup L_{d+1}$. We show that $\mathcal{F}'$ is a realizer of P.

For any $(x, y) \in \mathrm{Inc}(P)$. (a) If $x \in A'$ and $y \in A$, then $x > y$ in L_{d+1}; (b) If $x \in A$ and $y \in A'$, then $x > y$ in some L_j in $\mathcal{F}$ as $\mathcal{F}$ reverses $A \times A'$; (c) If $x, y \in A$ or $x, y \in A'$, then $x > y$ in either L_1 or L_{d+1}. Therefore $\mathcal{F}'$ is a realizer of P. $\square$

Note that for the range of p considered here, almost surely we have $\mathrm{Idim}(P) = \dim(P)$.

4.2.2 Matchings

A *matching of size* t in a bipartite poset $P = A \cup A'$ is a pair $\mathcal{M} = (T, T')$ where $T = \{a_1, a_2, \ldots, a_t\}$ and $T' = \{a_1', a_2', \ldots, a_t'\}$ are t-element subsets of A and A' respectively with $a_j \parallel a_j'$ for each $j = 1, 2, \ldots, t$. A matching is *maximal* if it is not properly contained in any other matchings. A matching is *maximum* if it has the largest size among all matchings. There is an obvious notion of a *complete* matching, i.e., a matching with $T = A$ and $T' = A'$.

The following basic result is due to Hiraguchi [25].

Proposition 4.2.2. *Let P be a poset and let $(x, y) \in \mathrm{Inc}(P)$. Then there exists a linear extension L of P such that:*

1. *If $z \in P - \{x\}$ and $x \parallel z$ in P, then $x > z$ in P.*

2. *If $w \in P - \{y\}$ and $y \parallel w$ in P, then $w > y$ in P.*

Proof. Note that since $(x, y) \in \mathrm{Inc}(P)$, we have $D(y) \cap U(x) = \emptyset$. Let $L = [D(y) < y < P - D[y] - U[x] < x < U(x)]$. It is then easy to check that L satisfies all requirements in Proposition 4.2.2. $\square$

When $P = A \cup A'$ is a bipartite poset and $(a, a') \in \mathrm{Inc}(A, A')$, we let $\mathcal{L}(a, a', P)$ denote the set of linear extensions of P satisfying the requirements of Proposition 4.2.2 for the pair (a, a').

Lemma 4.2.3. *Let $P = A \cup A'$ be a bipartite poset and let $\mathcal{M} = (T, T')$ be a maximal matching in P. If the size of $\mathcal{M}$ is t, then $\mathrm{Idim}(P) \leq t$.*

Proof. Let $T = \{a_1, a_2, \ldots, a_t\} \subseteq A$ and $T' = \{a'_1, a'_2, \ldots, a'_t\} \subseteq A'$ with $a_j \parallel a'_j$ for each $j = 1, 2, \ldots, t$. Then for each $j = 1, 2, \ldots, t$, let $L_j \in \mathcal{L}(a_j, a'_j, P)$. The family $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ is a realizer for $A \times A'$. $\square$

The following lemma is implicit in EKT and explicit in [5].

Lemma 4.2.4. *Let $P = A \cup A'$ be a bipartite poset with $\mathrm{Inc}(A, A') \neq \emptyset$. If $\mathrm{Idim}(P) = d$, then there exists a family of linear extensions $\mathcal{F} = \{L_1, L_2, \ldots, L_d\}$ which reverses $A \times A'$, subsets $T = \{a_1, a_2, \ldots, a_d\}$ and $T' = \{a'_1, a'_2, \ldots, a'_d\}$ of A and A', respectively, so that $\mathcal{M} = (T, T')$ is a matching in P and $L_j \in \mathcal{L}(a_j, a'_j, P)$ for every $j = 1, 2, \ldots, d$.*

Proof. Let $\mathcal{F} = \{L_1, L_2, \ldots, L_d\}$ be a family of linear extensions of P which reverses $A \times A'$. We modify the linear extensions in $\mathcal{F}$ according to the following iterative process: Suppose that $1 \leq j \leq d$ and that the procedure has already been applied to all L_i with $1 \leq i < j$.

Let a_j be the highest element of A in L_j and let a'_j be the lowest element of A' in L_j. Then $a_j \parallel a'_j$ in P and $a_j > a'_j$ in L_j; otherwise, we can delete L_j from $\mathcal{F}$ and obtain a new family of linear extensions of size $d-1$ which also reverses $A \times A'$.

Group all elements of A' which are incomparable with a_j in P but higher than a_j in L_j and put them in a block immediately under a_j. The order on these elements (if there are any) stays unchanged. Dually, group all elements of A which are incomparable with a'_j in P but lower than a'_j in L_j and put them in a block immediately over a'_j.

Finally, if $j < d$, move a_j to the bottom of L_i for all i with $j < i \leq d$. Dually, move a'_j to the top of L_i for all i with $j < i \leq d$. It is easy to see that when this iterative process completes, the modified family of linear extensions $\mathcal{F}$ and the appropriate matching $\mathcal{M} = (T, T')$ will be obtained. $\square$

4.3 Upper Bounds, Latin Rectangles and Euler Functions

One of the advantages of working with the parameter $\mathrm{Idim}(P)$ for bipartite posets is that it can be defined in terms of linear orders on one of the two sides. To be more precise, when $P = A \cup A'$ is a bipartite poset, and $\mathrm{Inc}(A, A') \neq \emptyset$, then $\mathrm{Idim}(P)$ is the least positive integer d for which there is a family $\mathcal{F} = \{L_1, L_2, \ldots, L_d\}$ of linear orders on A satisfying the following property:

Realizer Property. For every pair $(a, a') \in \mathrm{Inc}(A, A')$, there is some j with $1 \leq j \leq d$, so that $a' \parallel b$ in P for all $b \in A$ with $0 \leq h_j(b) < h_j(a)$.

Of course, this definition can also be applied in terms of linear orders on A'. Regardless, as we progress through the chapter, readers may note that in some cases, we will find it convenient to emphasize the one-sided formulation of dimension, while in other cases, the role of matchings and the symmetry of the two-sided perspective will be more important.

All the results in this first section will emphasize the one-sided definition of dimension. We begin with an elementary proof of Theorem 4.1.7. It differs from the original proof of the upper bound in Theorem 4.1.6 in that we want to move away from considering a family of independent events, and work instead with expected value.

Proof. Let $\epsilon > 0$ and $\log^{1+\epsilon} n/n < q \leq 1/2$. Set $m = qn/((2+\epsilon)\log n)$ and $t = n - m$. We will then show that, almost surely $\text{Idim}(P) \leq t$. To accomplish this, we set $r = t/m$ and construct a family $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ of linear orders on A as follows. First, let $T = \{a_1, a_2, \ldots, a_t\}$ be an arbitrary t-element subset of A. For each $j = 1, 2, \ldots, t$, we make a_j the highest element of L_j, i.e., we set $h_j(a_j) = 0$.

Let $M = A - T$, and let $\{x_1, x_2, \ldots, x_m\}$ be a labelling of the elements of M. Then for every $\alpha = 1, 2, \ldots, m$, we set $h_j(x_\alpha) = 1$ when $(\alpha - 1)r < j \leq \alpha r$. The ordering on the remaining $n - 2$ elements of A in L_j is arbitrary.

Let X be the random variable which counts the number of pairs $(a, a') \in A \times A'$ for which $a \parallel a'$ in P and there is no j with $1 \leq j \leq t$ for which $h_j(a) \leq 1$ and $a' \parallel b$ for every $b \in A$ with $0 \leq h_j(b) < h_j(a)$. Note that there are no such pairs when

$a \in T$. It follows that the expected value of X is given by:

$$\begin{aligned}
E[X] &= |A - T||A'| \cdot q(1-q)^r \\
&= mnq(1-q)^{t/m} \\
&< n^2 q e^{-qt/m} \\
&= n^2 e^{-qn/m} q e^q \\
&< n^2 e^{-qn/m} \\
&= e^{2\log n - (2+\epsilon)\log n} \\
&= e^{-\epsilon \log n} \\
&= n^{-\epsilon} \\
&< \epsilon.
\end{aligned}$$

By Markov's inequality, we have $\Pr(X \geq 1) \leq E[X] < \epsilon$, i.e., $X = 0$ almost surely. We note that when $X = 0$, the family $\mathcal{F}$ witnesses that $\mathrm{Idim}(P) \leq t$. Therefore almost surely we have $\mathrm{Idim}(P) \leq t$. $\square$

We note that the lower bound $q > (\log^{1+\epsilon} n)/n$ serves only to insure that m is large so that round off errors can be ignored.

On the one hand, at least for a very wide range of values of q, we will see that the simple construction in Theorem 4.1.7 is remarkably close to being best possible. Nevertheless, a delicate improvement can still be made by paying greater attention to the details of the construction. In fact, we will present two fundamentally different approaches, and we consider it surprising that ultimately, the two approaches yield the same end result. Our first approach carries on in the spirit of the preceding proof.

4.3.1 Generalized Latin Rectangles

Recall that when m and s are integers with $1 \leq s \leq m$, an $s \times m$ array (matrix) R is called a *latin rectangle* when (1) each row of R is a permutation of the integers in

$\{1, 2, \ldots, m\}$, and (2) the entries in each column of R are distinct. One of the classic results in combinatorial theory asserts that if $1 \leq s < m$, an $s \times m$ latin rectangle R can always be extended to a $(s+1) \times m$ latin rectangle.

Now let (m, r, s) be a triple of positive integers. An $s \times (rm)$ array R will be called an (m, r, s)-*generalized latin rectangle* (GLR) when the following conditions are met:

1. In each row of R, each symbol in $\{1, 2, \ldots, m\}$ occurs exactly r times.
2. In each column C of R, the s symbols in $\{1, 2, \ldots, m\}$ occurring in column C are distinct.
3. For each distinct pair $i, j \in \{1, 2, \ldots, m\}$, there is at most one column C in R for which i is below j in column C.

Note that when $r = 1$, the third requirement is not part of the traditional definition for a latin rectangle. However, it will be soon be clear why we want this additional restriction in place.

Here is an example of a $(9, 2, 3)$-GLR.

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 \\ 8 & 9 & 9 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 \\ 3 & 6 & 4 & 7 & 5 & 8 & 6 & 9 & 7 & 1 & 8 & 2 & 9 & 3 & 1 & 4 & 2 & 5 \end{bmatrix}$$

We note that it is impossible to extend this array to a $(9, 2, 4)$-GLR (as a result of Lemma 4.3.1). Nevertheless, we have the following natural extremal problem: For a pair (m, r) of positive integers, what is the largest s for which there is an (m, r, s)-GLR? For an upper bound, we have the following result.

Lemma 4.3.1. *Let m, r and s be positive integers. If there is an (m, r, s)-GLR, then $rs(s-1) \leq 2(m-1)$.*

Proof. Suppose that R is an (m, r, s)-GLR. There are rm columns in R and for each column C in R, there are $s(s-1)/2$ ordered pairs (i, j) where i is below j in column C. The last two conditions in the definition of an (m, r, s)-GLR force $rms(s-1)/2 \leq m(m-1)$, so that $rs(s-1) \leq 2(m-1)$. $\square$

Lower bounds on this extremal problem are more challenging, but the following bound will be sufficient for our purposes. Note that the proof mimics the argument for the classic result for latin rectangles.

Lemma 4.3.2. *Let m, r and s be positive integers. If $rs^2(s+1) < m$, then any (m, r, s)-GLR can be extended to an $(m, r, s+1)$-GLR.*

Proof. Let R be an arbitrary (m, r, s)-GLR. Naturally, we view R as a union of r latin rectangles, each of size $s \times m$. We show that it is possible to extend each of those r latin rectangles to $s+1$ rows by adding an appropriately chosen new row at the bottom, and the union of r new latin rectangles forms $(m, r, s+1)$-GLR.

Assume the first $k (< r)$ latin rectangles have been extended to $s+1$ rows. We show that it is possible to extend the $(k+1)$-th latin rectangle to $s+1$ rows while the following condition is satisfied: For each distinct pair $i, j \in \{1, 2, \ldots, m\}$, there is at most one column C of all r latin rectangles for which i is below j in column C.

For each $i = 1, 2, \ldots, m$, the symbol i appears in s columns of the $(k+1)$-th latin rectangle, and we cannot add i to the bottom of any of these columns. Counting all $rs+k$ appearances of symbol i in all r latin rectangles, there are exactly $ks+rs(s-1)/2$ other symbols which appear above i. If the other occurrences of these symbols are in distinct columns, then together, they would rule out at most $s[ks + rs(s-1)/2]$

other columns of the $(k+1)$-th latin rectangle. Since

$$\begin{aligned} s + s[ks + rs(s-1)/2] &= s[1 + ks + rs(s-1)/2] \\ &\le s[1 + (r-1)s + rs(s-1)/2] \\ &\le s[rs + rs(s-1)/2] \\ &= rs^2(s+1)/2 \\ &< m/2 \end{aligned}$$

we conclude that there are at least $m/2$ columns where symbol i could be added legally at the bottom.

If such extension does not exist, then by Hall's theorem, there would be a subset $S \subseteq \{1, 2, \ldots, m\}$ so that the number of columns of the $(k+1)$-th latin rectangle for which there is some symbol in S which could legally be added at the bottom is less than $|S|$. But since any symbol has at least $m/2$ legal choices for columns, it requires that $|S| > m/2$.

But we claim for *any* column C of the $(k+1)$-th latin rectangle, and any subset S of size at least $m/2$, there is at least one symbol in S which could be legally placed at the bottom of column C.

To see this, let i be one of the s symbols in column C. We consider the number of symbols which cannot be appended to the bottom of C. Clearly symbol i cannot be appended to the bottom of C. Counting all $rs + k$ appearances of symbol i in all r latin rectangles, there are exactly $ks + rs(s-1)/2$ other symbols which appear below i. If those symbols are all distinct for different symbol i in column C, then at most $s + s[ks + rs(s-1)/2]$ symbols would be ruled out. However,

$$s + s[ks + rs(s-1)/2] \le rs^2(s+1)/2 < m/2.$$

So more than $m/2$ symbols can be legally placed at the bottom of column C. This observation completes the proof of the lemma. $\square$

4.3.2 Applying Generalized Latin Rectangles

Let (m, r, s) be a triple of positive integers and suppose that R is an (m, r, s)-GLR. Set $t = rm$ and let $|A| = t + m$. We use R to construct a family $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ of linear orders on A.

First, let $T = \{a_j : 1 \leq j \leq t\}$ be an arbitrary t-element subset of A. Then label the elements in $M = A - T$ as $\{x_1, x_2, \ldots, x_m\}$. For each $j = 1, 2, \ldots, t$, we set $h_j(a_j) = 0$, i.e., a_j is the highest element of L_j. Also, if the symbol α appears in row i and column j in the array R, we set $h_j(x_\alpha) = i$.

These rules determine the highest $s + 1$ positions in L_j for each $j = 1, 2, \ldots, t$. The order of the remaining $n - s - 1$ elements of A in L_j is arbitrary.

Let X be the random variable that counts the number of pairs $(a, a') \in A \times A'$ with $a \parallel a'$ in P but there is no value of j with $1 \leq j \leq t$ for which $0 \leq h_j(a) \leq s$ and $a' \parallel b$ in P for all $b \in A$ with $0 \leq h_j(b) < h_j(a)$. As before, there are no such pairs when $a \in T$.

Fix an element $a \in A$ and let $\mathcal{L} = \{L_j : 0 \leq h_j(a) \leq s, 1 \leq j \leq t\}$. Based on the third property of an (m, r, s)-GLR, all elements $b \in A$ with $0 \leq h_j(b) < h_j(a), L_j \in \mathcal{L}$ are distinct. Therefore, the expected value of X is given by:

$$E[X] = |A - T||A'| \cdot q \prod_{i=1}^{s} (1 - q^i)^r = nmq \prod_{i=1}^{s} (1 - q^i)^r. \tag{2}$$

Same as before, an upper bound on the expected value of $\mathrm{Idim}(P)$ can then be obtained by analyzing conditions on the parameters which drive $E[X]$ to 0. But we postpone the analysis to Section 4.3.4.

4.3.3 Another Upper Bound Construction

In some sense, this first construction takes an "egalitarian approach": While there is no escaping that some set T of elements in A get differential treatment and are positioned at the very top of a linear order L in $\mathcal{F}$, all the remaining elements of A are treated in *exactly the same way.* But we may consider another, completely different construction, one which continues the pattern of treating certain elements of A in a preferential manner relative to others.

We consider a partition $M = M_1 \cup M_2 \cup \cdots \cup M_s$. In some sense, this partition divides elements of M into classes and elements in M_{i_1} will be positioned higher in our linear orders than elements in M_{i_2} when $1 \leq i_1 < i_2 \leq s$.

To be more concrete, for each $j = 1, 2, \ldots, t$, we construct a linear order L_j on A using the following rules:

Only the highest $s+1$ positions in L_j will be specified. The order on the remaining elements is arbitrary. As before, we make a_j the highest element of L_j. After that basic step, we place the following three additional constraints on the construction, noting that all except possibly the third constraint are easy to satisfy.

1. For each $i = 1, 2, \ldots, s$ and each $j = 1, 2, \ldots, t$, if $x \in M$ and $h_j(x) = i$, then $x \in M_i$.

2. For each $i = 1, 2, \ldots, s$ and each $x \in M_i$, there are $t/|M_i|$ different values of j with $h_j(x) = i$.

3. For each pair $1 \leq i_1 < i_2 \leq s$ and each pair $x \in M_{i_1}$, $y \in M_{i_2}$, there is at most one j with $1 \leq j \leq t$ so that $h_j(x) = i_1$ and $h_j(y) = i_2$.

The resulting family is a realizer for $A \times A'$ when the following modified realizer property is satisfied.

Modified Strong Realizer Property. For every pair $(a, a') \in \mathrm{Inc}(A, A')$ with $a \in M_i$, there is some integer j with $1 \leq j \leq t$ such that $h_j(a) = i$ and $a' \parallel b$ in P for all $b \in A$ with $0 \leq h_j(b) < i$.

Now let Y_i be the random variable that counts the number of pairs $(a, a') \in M_i \times A'$ for which $a \parallel a'$ in P and there is no j with $h_j(a) = i$ and $a' \parallel b$ for all $b \in A$ with

$0 \le h_j(b) < i$. Then, similar as before, the expected value of Y_i is given by

$$E[Y_i] = |M_i||A'| \cdot q(1-q^i)^{t/|M_i|} = |M_i|nq(1-q^i)^{t/|M_i|}.$$

Considering the form of the above expression of $E[Y_i]$, it makes sense to add the following constraint to the construction to achieve uniformity:

4. For each $i = 1, 2, \ldots, s$, $(1-q)^{t/|M_1|} = (1-q^i)^{t/|M_i|}$.

Now let $Y = Y_1 + Y_2 + \cdots + Y_s$. Then the expected value of Y is given by:

$$\begin{aligned} E[Y] &= \sum_{i=1}^{s} E[Y_i] \\ &= \sum_{i=1}^{s} |M_i|nq(1-q^i)^{t/|M_i|} \\ &= \sum_{i=1}^{s} |M_i|nq(1-q)^{t/|M_1|} \\ &= nq(1-q)^{t/|M_1|} \sum_{i=1}^{s} |M_i| \\ &= nmq(1-q)^{t/|M_1|}. \end{aligned}$$

4.3.4 Comparting Two Upper Bounds

Returning to our previous calculation for the expected value $E[X]$, we observe that:

$$\begin{aligned} E[X] &= nmq \prod_{i=1}^{s} (1-q^i)^r \\ &= nmq \prod_{i=1}^{s} \left((1-q)^{|M_i|/|M_1|}\right)^{t/m} \\ &= nmq \prod_{i=1}^{s} (1-q)^{t|M_i|/(m|M_1|)} \\ &= nmq(1-q)^{t(|M_1|+|M_2|+\cdots+|M_s|)/(m|M_1|)} \\ &= nmq(1-q)^{t/|M_1|} \\ &= E[Y]. \end{aligned}$$

So while it is intuitively clear that this second construction can be carried out for relatively large values of s, there is no reason to waste time with the effort, as it will lead to the same result. As we have commented previously, we find it surprising that these two approaches, which we consider quite different in spirit, lead to *exactly* the same place. And this observation is a major part of our faith in Conjecture 4.1.9.

We now return to the task of analyzing the Equation 2. Let ϵ be any small positive constant and let

$$m = \frac{n(\log(1/\phi(q))}{(2+\epsilon)\log n}.$$

Then set $t = n - m$. Recall that $\phi(q)$ is the Euler function:

$$\phi(q) = \prod_{i=1}^{\infty}(1-q^i).$$

We show that almost surely, $\mathrm{Idim}(P) \leq t$ by showing that $E[X] < \epsilon$, provided that n is sufficiently large. Note that when n is large, we certainly have the inequality $t > (1-\epsilon/10)n$. Also, using Lemma 4.3.2, with $r = t/m$, we can take

$$s = \sqrt[3]{m/r} = \sqrt[3]{m^2/t} > \sqrt[3]{m^2/n} > n^{1/4}$$

Since $q \leq 1/2$, it then follows easily that when n is large, we have

$$\begin{aligned} \log\bigl(1/\prod_{i=1}^{s}(1-q^i)\bigr) &> (1-\epsilon/10)\log\bigl(1/\prod_{i=1}^{\infty}(1-q^i)\bigr) \\ &= (1-\epsilon/10)\log(1/\phi(q)) \end{aligned}$$

Taking logarithms of Equation 2, we see that

$$\begin{aligned}
\log E[X] &= \log\bigl(nmq\prod_{i=1}^{s}(1-q^i)^r\bigr) \\
&< 2\log n + \frac{t}{m}\log\bigl(\prod_{i=1}^{s}(1-q^i)\bigr) \\
&= 2\log n - \frac{t}{m}\log\bigl(1/\prod_{i=1}^{s}(1-q^i)\bigr) \\
&= 2\log n - \frac{t}{n}\frac{\log\bigl(1/\prod_{i=1}^{s}(1-q^i)\bigr)}{\log(1/\phi(q))}(2+\epsilon)\log n \\
&< 2\log n - (1-\epsilon/10)(1-\epsilon/10)(2+\epsilon)\log n \\
&< -\epsilon\log n/2.
\end{aligned}$$

This completes the proof of Theorem 4.1.8.

4.4 Improved Lower Bounds on Dimension

As we begin this section, we comment that the symbols R, s and r will be used here in a manner consistent with EKT, and they will no longer have any connection with material from the preceding section. Also, following EKT, our treatment of lower bounds will require a second variation on dimension, and this parameter is called *short dimension*. It requires some additional notation and terminology, starting with the notion of a “mixing threshold”.

When s is a positive integer with $s \leq n$, we say a bipartite poset $P = A \cup A'$ is *s-mixed* if for all subsets $U \subseteq A$, $U' \subseteq A'$ with $|U| = |U'| = s$, there is a pair $(u, u') \in U \times U'$ with $u < u'$ in P. The following elementary lemma is proved in EKT.

Lemma 4.4.1. *If $\lim_{n\to\infty} pn = \infty$ and $\lfloor 2\log pn/p\rfloor \leq s \leq n$, then in the probability space $\Omega(n,p)$, almost surely, P is s-mixed.*

When $q \leq 1/2$, this lemma, while entirely correct, can be tightened. Here is an elementary update.

Lemma 4.4.2. *Let $1/n^{1-\epsilon} < q \leq 1/2$ and let $s = (2+\epsilon)\log n/\log(1/q)$. Then in the probability space $\Omega(n,p)$, almost surely, P is s-mixed.*

Proof. Note that since $q > n^{-1+\epsilon}$, $s \geq 2$. Let X count the number of pairs (U, U') where U and U' are s-element subsets of A and A' respectively, with $u \parallel u'$ in P for all $(u, u') \in U \times U'$. Then the expected value of X is given by:

$$E[X] = \binom{n}{s}^2 q^{s^2} < n^{2s} q^{s^2} = n^{2s}(1/q)^{-s^2}.$$

It follows that

$$\log(E[X]) < s(2\log n - s\log(1/q)) = -s\epsilon \log n.$$

So that $E[X] < n^{-\epsilon s} \to 0$. This shows that almost surely, P is s-mixed. $\square$

So for the remainder of this section, the value of s is fixed at $s = (2+\epsilon)\log n/\log(1/q)$. We note that when $q \leq n^{-1/4}$, s is in fact at most 9.

4.4.1 Short Families, Short Realizers and Short Dimension

Let $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ be a family of linear extensions of a bipartite poset $P = A \cup A'$ which reverses $A \times A'$. Assume further that P is s-mixed. For each $j = 1, 2, \ldots, t$, consider the 4-tuple $(B_j, \sigma_j, B'_j, \sigma'_j)$ where:

1. B_j is the subset of A consisting of the s highest elements of A in L_j and σ_j is the restriction of L_j to B_j; and
2. B'_j is the subset of A' consisting of the s lowest elements of A' in L_j and σ'_j is the restriction of L_j to B'_j.

It follows that for every subset $\mathcal{S} \subseteq A \times A'$, the following property is satisfied:

Short Realizer Property. For every pair $(a, a') \in \mathcal{S}$, if $a \parallel a'$ in P, then there is some j with $1 \leq j \leq t$ for which one of the following two statements holds:

1. $a \in B_j$ and $a' \parallel b$ in P for every $b \in B_j$ which is higher than a in σ_j.

2. $a' \in B'_j$ and $a \parallel b'$ in P for every $b' \in B'_j$ which is lower than a' in σ'_j.

In the discussion to follow, we will refer to a family $\Sigma = \{(B_j, \sigma_j, B'_j, \sigma'_j) : 1 \leq j \leq t\}$ as a *short family* when for all $j = 1, 2, \ldots, t$, B_j and B'_j are s-element subsets of A and A', respectively, and σ_j and σ'_j are linear orders on B_j and B'_j, respectively. Note that we do not restrict our attention to short families which arise from families of linear extensions of P.

When $\mathcal{S} \subseteq A \times A'$, we say that Σ is a *short realizer for* $\mathcal{S}$ when the Short Realizer Property specified above is satisfied. We then define the *short dimension* of a bipartite poset $P = A \cup A'$, denoted $\mathrm{sdim}(P)$, as the least d for which there is a short family $\Sigma = \{(B_j, \sigma_j, B'_j, \sigma'_j) : 1 \leq j \leq d\}$ which is a short realizer for $A \times A'$.

There is a natural connection between dimension and short dimension.

Lemma 4.4.3. *If $P = A \cup A'$ is s-mixed, then* $\mathrm{sdim}(P) \leq \mathrm{Idim}(P)$.

Proof. Let $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ be a realizer which witnesses $\mathrm{Idim}(P) = t$. We consider a pair $(a, a') \in A \times A'$ with $a \parallel a'$ in P. Since $a \parallel a'$ in P, there is some L_j in $\mathcal{F}$ so that $a > a'$ in L_j. Same as before, we define B_j be the subset of A consisting of the s highest elements of A in L_j and σ_j be the restriction of L_j to B_j; and B'_j be the subset of A' consisting of the s lowest elements of A' in L_j and σ'_j be the restriction of L_j to B'_j.

Note that we have either $a \in B_j$ or $a' \in B'_j$. Suppose not and $a \notin B_j$ and $a' \notin B'_j$. Then in L_j we have $B_j > a > a' > B'_j$, which implies $u \parallel u'$ for all $(u, u') \in B_j \times B'_j$. This contradicts the fact that P is s-mixed.

We assume, without loss of generality, that $a \in B_j$. As $a > a'$ in L_j, we must have $a' \parallel b$ in P for every $b \in B_j$ which is higher than a in σ_j. Therefore the short family $\Sigma = \{(B_j, \sigma_j, B'_j, \sigma'_j) : 1 \leq j \leq t\}$ we constructed, based on $\mathcal{F}$, is in fact a short realizer of P. Hence $\mathrm{sdim}(P) \leq \mathrm{Idim}(P)$. $\square$

Now we fix an arbitrary short family $\Sigma = \{(B_j, \sigma_j, B'_j, \sigma'_j) : 1 \leq j \leq t\}$ and consider the event $R = R(\Sigma)$ that Σ is a short realizer for $A \times A'$. Following EKT, if we can find an upper bound p_0 of $\Pr(R)$ for all Σ, then we will have a lower bound on $\mathrm{sdim}(P)$. To see this, suppose we have an upper bound p_0 of $\Pr(R)$. Note that the number of short families is relatively small. In fact it is at most $(P(n,s))^{2t}$ as we count all s-permutation of n objects. Since $(P(n,s))^{2t} < n^{2st}$, we can be assured that almost surely $\mathrm{sdim}(P) > t$ provided $p_0 n^{2st} < \epsilon$.

As noted in EKT, in finding an upper bound p_0 on $\Pr(R)$, we can restrict our attention to short families satisfying the following two conditions.

1. There is a t-element subset $T = \{a_1, a_2, \ldots, a_t\} \subseteq A$ so that if $1 \leq j_1, j_2 \leq t$, then a_{j_1} is the highest element of B_{j_1} in σ_{j_1}, and if $j_2 \neq j_1$, then $a_{j_1} \notin B_{j_2}$.

2. There is a t-element subset $T' = \{a'_1, a'_2, \ldots, a'_t\} \subseteq A'$ so that if $1 \leq j_1, j_2 \leq t$, then a'_{j_1} is the lowest element of B'_{j_1} in σ'_{j_1}, and if $j_2 \neq j_1$, then $a'_{j_1} \notin B'_{j_2}$.

Accordingly, we will set $M = A - T$ and $M' = A' - T'$ and let $m = |M| = |M'|$.

The reason that we can focus only on those short families is that suppose a_j is the highest element of B_j in σ_j for some j, all pairs $(a, a') \in A \times A'$ with $a = a_j$ and $a \parallel a'$ can be reversed by the short realizer property. Therefore, as we are looking for an upper bound p_0 of $\Pr(R)$, there is no advantage to placing a_j in other B_i's with $i \neq j$. If a_j appears in some B_i with $i \neq j$, we can simply remove a_j from B_i, push all elements below a_j in σ_i up one position, and add a new element to B_i as the bottom of σ_i. Note that this can only increase the probability that a given short family is a short realizer.

In the text follows, we will assume Σ satisfies the above two conditions.

Recall that $R = R(\Sigma)$ is the event that Σ is a short realizer for $A \times A'$. To estimate the probability that R will occur, we write R into intersections of simpler events. Let

$R(a, a')$ denote the event that Σ realizes pair (a, a'). Then we have

$$R = \bigcap_{(a,a') \in A \times A'} R(a, a').$$

Note that we have either $a \parallel a'$, or $a < a'$. We let $(a < a')$ be the event that $a < a'$ in P. Let $Q_j(a, a')$ denote the event that pair (a, a') is realized via σ_j, i.e., $a \in B_j$ and $a' \parallel b$ in P for every $b \in B_j$ which is higher than a in σ_j. Dually, let $Q'_j(a, a')$ denote the event that pair (a, a') is realized via σ'_j, i.e., $a' \in B'_j$ and $a \parallel b'$ in P for every $b' \in B'_j$ which is lower than a' in σ'_j. Then Let $Q(a, a') = \cup_{j=1}^{t} Q_j(a, a')$ and $Q'(a, a') = \cup_{j=1}^{t} Q'_j(a, a')$. Then we have

$$R(a, a') = (a < a') \cup Q(a, a') \cup Q'(a, a').$$

Note that $Q(a, a')$ and $Q'(a, a')$ are defined independent of the relation between a and a'. Also, $Q(a, a')$ concerns the relations between a' and b where $a \neq b$, while $Q(a, a')$ concerns the relations between a and b' where $a' \neq b'$. Hence all three events $(a < a')$, $Q(a, a')$ and $Q'(a, a')$ are mutually independent. Therefore

$$\neg R(a, a') = \neg(a < a') \cap \neg Q(a, a') \cap \neg Q'(a, a')$$

And

$$\begin{aligned} \Pr(\neg R(a, a')) &= \Pr(\neg(a < a')) \Pr(\neg Q(a, a')) \Pr(\neg Q'(a, a')) \\ &= q \Pr(\neg Q(a, a')) \Pr(\neg Q'(a, a')) \end{aligned}$$

Now we want to estimate the probability of $\neg Q(a, a')$ and $\neg Q'(a, a')$. As they are symmetric, we will focus only on $\Pr(\neg Q(a, a'))$. By definition, we have $\neg Q(a, a') = \cap_{j=1}^{t} \neg Q_j(a, a')$. Note that the events $Q_j(a, a')$, $1 \leq j \leq t$ are not necessarily independent. However, we shall see that they are positively correlated. To prove this, we need the following Ahlswede and Daykin [1] inequality, which was first proved by Kleitman [31].

Proposition 4.4.4. *If P on ground set X is the family of subsets of a finite set ordered by inclusion, and U_1 and U_2 are up sets of X, then $|U_1|/|X| \le |U_1 \cap U_2|/|U_2|$.*

Lemma 4.4.5. *The events $\neg Q_j(a, a')$, $1 \le j \le t$ are positively correlated.*

Proof. For each $2 \le i \le t$, let $E_i = \cap_{j=1}^{i-1} \neg Q_j(a, a')$. Using conditional probability, we have

$$\begin{aligned}\Pr(\neg Q(a, a')) &= \Pr[\cap_{j=1}^{t} \neg Q_j(a, a')] \\ &= \Pr(\neg Q_1(a, a')) \Pr(\neg Q_2(a, a') \mid E_2) \cdots \Pr(\neg Q_t(a, a') \mid E_t)\end{aligned}$$

For $P, P' \in \Omega(n, p)$, we let $P \subseteq P'$ if and only if for all $(a, a') \in A \times A'$, $a < a'$ in P implies $a < a'$ in P'. For each $(a, a') \in A \times A'$ and $1 \le j \le t$, note that event $\neg Q_j(a, a')$ is an up set in $\Omega(n, p)$. For each $2 \le i \le t$, as E_i is the intersection of up sets, E_i is also an up set in $\Omega(n, p)$. Applying Proposition 4.4.4, we have

$$\begin{aligned}\Pr(\neg Q(a, a')) &= \Pr(\neg Q_1(a, a')) \Pr(\neg Q_2(a, a') \mid E_2) \cdots \Pr(\neg Q_t(a, a') \mid E_t) \\ &\ge \Pr(\neg Q_1(a, a')) \Pr(\neg Q_2(a, a')) \cdots \Pr(\neg Q_t(a, a')) \\ &= \prod_{j=1}^{t} \Pr(\neg Q_j(a, a'))\end{aligned}$$

Therefore the events $\neg Q_j(a, a')$, $1 \le j \le t$ are positively correlated. $\square$

Following Lemma 4.4.5, we have

$$\begin{aligned}\Pr(\neg Q(a, a')) &\ge \prod_{j=1}^{t} \Pr(\neg Q_j(a, a')) \\ &\ge \prod_{j=1}^{t} (1 - q^{\sigma_j(a)})\end{aligned}$$

The same formula also applies to $\Pr(\neg Q'(a, a'))$.

$$\begin{aligned}\Pr(\neg Q'(a, a')) &\ge \prod_{j=1}^{t} \Pr(\neg Q'_j(a, a')) \\ &\ge \prod_{j=1}^{t} (1 - q^{\sigma'_j(a')})\end{aligned}$$

Therefore

$$\begin{aligned}\Pr(\neg R(a,a')) &= q \Pr(\neg Q(a,a')) \Pr(\neg Q'(a,a')) \\ &\geq q \prod_{j=1}^{t} (1 - q^{\sigma_j(a)})(1 - q^{\sigma'_j(a')})\end{aligned}$$

Recall that our goal is to find an upper bound of $\Pr(R)$. However, we can not estimate $\Pr(R)$ directly, as events $R(a,a')$ where $(a,a') \in A \times A'$ are not independent. Moreover, events $R(a,a')$ where $(a,a') \in A \times A'$ are not negatively correlated, i.e., we do not know if $\Pr(R) \leq \prod_{(a,a') \in A \times A'} \Pr(R(a,a'))$. Note that we do need a similar inequality, as we are looking for an upper bound of $\Pr(R)$.

In order to overcome the obstacle, we shall restrict our attention to a smaller set $\mathcal{S}$, where $\mathcal{S}$ is a proper subset of $A \times A'$. It will be clear later that the events $R(a,a')$ "restricted" on $\mathcal{S}$ are independent.

Let $X \subseteq M$ and $X' \subseteq M'$. For $(a,a') \in X \times X'$, let $R(a,a'/X,X')$ be the event that Σ realizes pair (a,a') over X and X'. (Note that we need not to worry about pair (a,a') with $a \in T$ or $a' \in T'$, as all such pairs are realized automatically per the short realizer property.) To be more precise, we define

$$R(a,a'/X,X') = (a < a') \cup Q(a,a'/X) \cup Q'(a,a'/X').$$

where

$$\begin{aligned}Q(a,a'/X) &= \bigcup_{j=1}^{t} Q_j(a,a'/X); \\ Q'(a,a'/X') &= \bigcup_{j=1}^{t} Q'_j(a,a'/X').\end{aligned}$$

Here $Q_j(a,a'/X)$ is the event that pair (a,a') is realized via σ_j over X, i.e., $a \in B_j$ and $a' \parallel b$ in P for every $b \in \{a_j\} \cup (X \cap B_j)$ which is higher than a in σ_j. Dually, $Q'_j(a,a'/X')$ is the event that pair (a,a') is realized via σ'_j over X', i.e., $a' \in B'_j$ and $a \parallel b'$ in P for every $b' \in \{a'_j\} \cup (X' \cap B'_j)$ which is lower than a' in σ'_j.

Note that since $Q_j(a,a') \subseteq Q_j(a,a'/X)$ and $Q'_j(a,a') \subseteq Q'_j(a,a'/X')$, we have $Q(a,a') \subseteq Q(a,a'/X)$ and $Q'(a,a') \subseteq Q'(a,a'/X')$. Therefore $R(a,a') \subseteq R(a,a'/X,X')$.

We mention that there is a natural generalization of the height function that we defined earlier in Section 4.2. When $X \subseteq M$ and $1 \leq j \leq t$, $h_j(a/X)$, the *height of* a *over* X *in* σ_j, is defined as:

$$h_j(a/X) = \begin{cases} |\{b \in \{a_j\} \cup (X \cap B_j) : b > a \text{ in } \sigma_j\}| & \text{if } a \in X \cap B_j \\ \infty & \text{otherwise} \end{cases}$$

Similarly, when $X' \subseteq M'$ and $1 \leq j \leq t$, the *height of* a' *over* X' *in* σ'_j is defined as:

$$h'_j(a'/X') = \begin{cases} |\{b' \in \{a'_j\} \cup (X' \cap B'_j) : b' < a' \text{ in } \sigma'_j\}| & \text{if } a' \in X' \cap B'_j \\ \infty & \text{otherwise} \end{cases}$$

For $a \in X$ and $1 \leq k \leq s-1$, let the *k-multiplicity* of a over X be $u_k(a/X) = |\{j : h_j(a/X) = k\}|$. Dually, the *k-multiplicity* of a' over X' be $u'_k(a'/X') = |\{j : h'_j(a'/X') = k\}|$.

Next, we define a weight function w. Let the *weight of* a *over* X be $w(a/X)$ where

$$w(a/X) = \sum_{i=1}^{s-1} u_i(a/X) 2^{1-i}.$$

The *weight of* a' *over* X' is defined dually.

Before we start to estimate the probability of $R(a, a'/X, X')$, we need the following elementary proposition.

Proposition 4.4.6. *Let* $0 < q \leq 1/2$. *Then for every integer* $i \geq 1$ *and every positive real number* x, *we have*

$$(1 - q^i)^x < (1 - q^{i+1})^{2x}.$$

Proof. The stated inequality is equivalent to $1 - q^i < (1 - q^{i+1})^2$. But this holds if and only if $1 - q^i < 1 - 2q^{i+1} + q^{2i+2}$, which is equivalent to $q^i(1 - 2q + q^{i+2}) > 0$. Clearly the inequality holds for $0 < q \leq 1/2$. □

Now we shall focus on $\Pr(R(a, a'/X, X'))$.

Lemma 4.4.7. $\Pr(R(a, a'/X, X')) \leq 1 - q(1 - q)^{w(a/X) + w(a'/X')}$.

Proof. Similar to the analysis of $R(a,a')$, events $(a<a')$, $Q(a,a'/X)$ and $Q'(a,a'/X')$ are mutually independent. Recall that

$$R(a,a'/X,X') = (a<a') \cup Q(a,a'/X) \cup Q'(a,a'/X').$$

Taking the negation of the above equation, we have

$$\neg R(a,a'/X,X') = (a \parallel a') \cap \neg Q(a,a'/X) \cap \neg Q'(a,a'/X').$$

Therefore,

$$\Pr(\neg R(a,a'/X,X')) = q \Pr(\neg Q(a,a'/X)) \Pr(\neg Q'(a,a'/X')).$$

Similar to the proof of Lemma 4.4.5, Proposition 4.4.4 also implies events $\neg Q_j(a,a'/X)$, $1 \le j \le t$ are positively correlated. We have

$$\begin{aligned}
\Pr(\neg Q(a,a'/X)) &= \Pr[\cap_{j=1}^{t} \neg Q_j(a,a'/X)] \\
&\ge \prod_{j=1}^{t} \Pr(\neg Q_j(a,a'/X)) \\
&= \prod_{j=1}^{t} (1-\Pr(Q_j(a,a'/X))) \\
&= \prod_{j=1}^{t} (1-q^{h_j(a/X)}) \\
&= \prod_{k=1}^{s-1} (1-q^k)^{u_k(a/X)} \\
&\ge \prod_{k=1}^{s-1} (1-q)^{u_k(a/X)2^{1-k}} \\
&= (1-q)^{w(a/X)}
\end{aligned}$$

The second last line in this computation holds in view of Proposition 4.4.6. Dually we have $\Pr(\neg Q'(a,a'/X')) \ge (1-q)^{w(a'/X')}$. Hence $\Pr(R(a,a'/X,X')) \le 1-q(1-q)^{w(a/X)+w(a'/X')}$. □

Lemma 4.4.8. *Let $(X,X') \subseteq (M,M')$. The average value of $w(a,X)$, for $a \in X$, is at most $2t/|X|$, and the average value of $w(a',X')$, for $a' \in X'$, is at most $2t/|X|$.*

Proof. The total weight of all $a \in X$ is

$$\begin{aligned}
\sum_{a \in X} w(a/X) &= \sum_{a \in X} \sum_{k=1}^{s-1} u_k(a/X) 2^{1-k} \\
&= \sum_{a \in X} \sum_{k=1}^{s-1} 2^{1-k} |\{j : h_j(a/X) = k\}| \\
&\leq \sum_{j=1}^{t} \sum_{k=1}^{s-1} 2^{1-k} \\
&< \sum_{j=1}^{t} 2 \\
&= 2t
\end{aligned}$$

Therefore the average value of $w(a, X)$, for $a \in X$, is at most $2t/|X|$. An analogous statement holds for the subset X'. □

For each $(a, a') \in X \times X'$, recall that $Q_j(a, a'/X)$ is the event that pair (a, a') is realized via σ_j over X, i.e., $a \in B_j$ and $a' \parallel b$ in P for every $b \in \{a_j\} \cup (X \cap B_j)$ which is higher than a in σ_j. We shall pay close attention to all such b's associated with the same a'. The *witness set of* a *over* X is defined as $W(a/X) = \{b \in X : a \leq b \text{ in some } \sigma_j\}$. Dually, the *witness set of* a' *over* X' is $W'(a'/X') = \{b' \in X' : a' \geq b' \text{ in some } \sigma'_j\}$. Then, the *witness set of* (a, a') *over* (X, X') is defined as

$$W(a, a'/X, X') = (\{a\} \times W'(a'/X')) \cup (W(a/X) \times \{a'\}).$$

We note that if the witness sets $\{W(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ are pairwise disjoint, then events $\{R(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ are mutually independent.

Now we want to construct a set $\mathcal{S} \in A \times A'$ so that set $\mathcal{S}$ is not too small compare to $A \times A'$ and $\{W(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ are pairwise disjoint. The set $\mathcal{S}$ will have form

$$\mathcal{S} = \cup_{i=1}^{r} S_i \times S'_i,$$

where $r = 2st^2/m$ is a constant. We also require $S_i \cap S_j = \emptyset$ and $S'_i \cap S'_j = \emptyset$ for all $1 \leq i < j \leq r$.

Lemma 4.4.9. *If S_i, $1 \le i \le r$, satisfies the following conditions,*

1. *For $1 \le i \le r$, if $a, b \in S_i$ and $a \ne b$, then $W(a/X(a)) \cap W(b/X(b)) = \emptyset$, and so is $a', b' \in S'_i$.*

2. *For $1 \le i < j \le r$, if $a \in S_i$ and $b \in S_j$, then $a \notin W(b/X(b))$, and so is $a' \in S'_i$ and $b' \in S'_j$.*

Then $\{W(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ are pairwise disjoint.

Proof. Suppose there exist two distinct pairs $(a, a'), (b, b') \in \mathcal{S}$ so that the intersection of $W(a, a'/X(a), X'(a'))$ and $W(b, b'/X(b), X'(b'))$ is not empty. Let (x, x') be the element in the intersection. Without loss of generality, we may assume $a \ne b$. Condition (1) implies that a and b belong to different sets. Let $a \in S_i$, $b \in S_j$ where, without loss of generality, $1 \le i < j \le r$. As $a \notin W(b/X(b))$, we must have $(x, x') \in (\{b\} \times W'(b'/X'(b')))$ and $(x, x') \in (W(a/X(a)) \times \{a'\})$. Therefore we have $a' \in W(b'/X(b'))$, which contradicts condition (2). This contradiction completes the proof of Lemma 4.4.9. □

In view of Lemma 4.4.9, we can now construct sets S_i and S'_i independently. In particular, we shall focus on sets S_i, and the construction on S'_i will be similar. Recall that $M = A - T$ and $m = |M| = n - t$. On average, each element in M appears $(s-1)t/m < st/m$ times in Σ. Hence there exists $X \subseteq M$ so that each element in X appears at most $2st/m$ times in Σ and $X \ge |M|/2 = m/2$.

For $1 \le i \le r$, S_i is a subset of X, and we require $|S_i| = m^2/16s^2t$. Now we shall construct S_i inductively. Suppose we have constructed $S_1, S_2, \ldots, S_k$ and the first l elements in S_{k+1} with $0 \le k < r$ and $0 \le l < m^2/16s^2t$. Let

$$Y = X - \cup_{i=1}^{k} S_i - \cup\{B_j : a \in B_j \text{ for some } a \in S_{k+1}\}.$$

We have

$$
\begin{aligned}
|Y| &\geq |X| - k|S_i| - s\frac{2st}{m}|S_i| \\
&\geq \frac{m}{2} - \frac{2s^2t}{m}\frac{m^2}{16s^2t} - \frac{2s^2t}{m}\frac{m^2}{16s^2t} \\
&= \frac{m}{2} - \frac{m}{8} - \frac{m}{8} \\
&= \frac{m}{4}
\end{aligned}
$$

By Lemma 4.4.8, the average value of $w(b, Y)$, for $b \in Y$, is at most $2t/|Y|$. Hence there exists $a \in Y$ such that $w(a, Y) \leq 2t/|Y| \leq 8t/m$. Let a be the $(l+1)$-th element in S_{k+1}, and let $X(a) = Y$. Note that if b is one of the first l elements in S_{k+1}, then $W(a/X(a)) \cap W(b/X(b)) = \emptyset$ as we exclude all B_j where $b \in B_j$ from Y. If $b \in S_i$ where $1 \leq i \leq k$, then $b \notin X(a)$ as S_i is excluded from Y. Hence $b \notin W(a/X(a))$. By Lemma 4.4.9, $\{W(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ are pairwise disjoint. Therefore events $\{R(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ are mutually independent.

Now we have an upper bound of $\Pr(R)$.

Lemma 4.4.10. $\Pr(R) \leq \left(1 - q(1-q)^{16t/m}\right)^{m^3/128s^2t}$.

Proof. This is now a straightforward calculation. Note that

$$R = \cap_{(a,a') \in A \times A'} R(a, a') \subseteq \cap_{(a,a') \in \mathcal{S}} R(a, a') \subseteq \cap_{(a,a') \in \mathcal{S}} R(a, a'/X(a), X'(a')).$$

We have

$$\Pr(R) \leq \Pr[\cap_{(a,a') \in \mathcal{S}} R(a, a'/X(a), X'(a'))].$$

As events $\{R(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ are mutually independent, we have

$$\Pr(R) \leq \prod_{(a,a') \in \mathcal{S}} \Pr(R(a, a'/X(a), X'(a'))).$$

By Lemma 4.4.7,

$$
\begin{aligned}
\Pr(R) &\leq \prod_{(a,a')\in\mathcal{S}} 1 - q(1-q)^{w(a/X)+w(a'/X')} \\
&\leq \prod_{(a,a')\in\mathcal{S}} 1 - q(1-q)^{16t/m} \\
&= \left(1 - q(1-q)^{16t/m}\right)^{r|S_i|^2} \\
&= \left(1 - q(1-q)^{16t/m}\right)^{m^3/128s^2t}.
\end{aligned}
$$

□

4.4.2 Proof of Lower Bounds

In this subsection, we will prove the two lower bounds in Theorems4.1.10 and 4.1.11. All of this will be done simply by analyzing the following inequality:

$$n^{2st}\left(1 - q(1-q)^{16t/m}\right)^{m^3/128s^2t} < \epsilon.$$

As we stated before, the number of short families is relatively small. In fact it is at most n^{2st}. If the parameters m and t are chosen so that the above inequality holds, then we can be assured that almost surely $\mathrm{Idim}(P) \geq \mathrm{sdim}(P) > t$.

As the argument progresses, we will have developed a sequence of inequalities, all of the form: $x < \epsilon y$ where $x = x(n, q, t, m, s)$ and $y = y(n, q, t, m, s)$. We find it convenient to work with the following short hand notation. Rather than considering inequalities, we write "equations" using symbol "$\doteq$". But now the "equation" $x \doteq y$ really means $\lim_{n\to\infty} x/y = 0$. With this convention, the three "equations" $x \doteq y$, $10x \doteq y$ and $y \doteq 10x$ all have the same meaning. Note that when x and y both tend to infinity, then $x \doteq y$ has the same meaning as $\log x \doteq \log y$.

With these comments in mind, we then analyze the following "equation", which we refer to as the Master Equation (ME).

$$n^{2st}\left(1 - q(1-q)^{16t/m}\right)^{m^3/128s^2t} \doteq 1.$$

Proof of Theorem 4.1.10. We consider the situation where C is a positive constant, $m = Cqn/\log n$, and $t = n - m$. We assume further that $m \to \infty$. Note that since $q \leq 1/2$, $m = o(n)$ and $t = (1 - o(1))n$.

With the given values for m and t, we note that:

$$1 - q(1-q)^{16t/m} = 1 - qe^{-16qt/m} = 1 - qn^{-16/C}.$$

So the ME becomes:

$$n^{2st}(1 - qn^{-16/C})^{m^3/128s^2t} \doteq 1.$$

Therefore

$$n^{2st}e^{-qm^3/128s^2tn^{16/C}} \doteq 1.$$

So taking logarithms (and ignoring the constant), we obtain:

$$st\log n \doteq \frac{qm^3}{s^2tn^{16/C}}.$$

Substituting in for m, we then have:

$$s^3n^{16/C}\log^4 n \doteq q^4C^3n.$$

In view of the known value of s, we can take:

$$n^{16/C}\log^7 n \doteq q^4C^3n\log^3 1/q. \tag{3}$$

Note that if $q \leq n^{-1/4+4/C-\epsilon}$, then $q^4n \leq n^{16/C-4\epsilon}$, which cannot be held in view of the above "equation". This explains the threshold value of $q = n^{-1/4}$ in our lower bounds. However, if we keep $q \geq n^{-1/4+\epsilon}$, then the above "equation" holds when $C > 4/\epsilon$ (the extra slack is there to absorb the $\log^7 n$ term and the $\log^3 1/q$ term). With this observation, the proof of Theorem 4.1.10 is now complete. □

Proof of Theorem 4.1.11. We return to the analysis of the ME and take C as a function of n and q. We shall focus on the case where $\log^{1+\epsilon} n/n < q \leq n^{-1/4}$. Note that when $tq/m \to \infty$, i.e, when $C = o(\log n)$, $q^4C^3n\log^3 1/q \leq C^3\log^3 1/q < C^3\log^3 n$.

In view of "equation" (3), we then have $n^{16/C}\log^7 n = C^3\log^3 n$, which is clearly false for $C = o(\log n)$. Therefore, we must have $C = \Omega(\log n)$. So there exists a constant c so that $C \geq 16c\log n$.

Since $q \leq 1/2$, we have $m = o(n)$ and $t = (1 - o(1))n$. With the given values for m and t, we note that:

$$1 - q(1-q)^{16t/m} = 1 - qe^{-16qt/m} \leq 1 - qe^{-1/c}.$$

So the ME becomes:

$$n^{2st}(1 - qe^{-1/c})^{m^3/128s^2t} \doteq 1.$$

Therefore

$$n^{2st}e^{-qe^{-1/c}m^3/128s^2t} \doteq 1.$$

So taking logarithms (and ignoring the constant), we obtain:

$$st\log n \doteq \frac{qm^3}{s^2t}.$$

Substituting in for m, we then have:

$$s^3\log n \doteq C^3q^4n.$$

Note that when $q \leq n^{-1/4}$, s is a small constant, so our equation becomes:

$$\log n \doteq C^3q^4n.$$

The above "equation" holds if $C^3q^4n = \log^{1+\epsilon} n$, i.e., $C = n^{-1/3}q^{-4/3}\log^{(1+\epsilon)/3} n$. It follows that almost surely, $\mathrm{Idim}(P) > n - Cqn/\log n = n - n^{2/3}q^{-1/3}\log^{(\epsilon-2)/3} n$. This completes the proof of Theorem 4.1.11. $\square$

An Aside on the EKT Lower Bound. In hindsight, most of the complexity in the original EKT lower bound argument for Theorem 4.1.1 resulted from handling the case when $p = o(1/\log n)$. In this range, all of the nice structural properties associated with our GLR constructions vanish since the subsets T and T' become very small fractions of A and A', respectively.

4.5 An Extremal Problem and Its Implications for Random Posets

Recall a classic theorem of Hiraguchi [25] which asserts that $\dim(P) \leq |P|/2$, when $|P| \geq 4$. Combining results of [30] and [10], it is known that if $d \geq 4$ and $|P| \leq 2d+1$, then $\dim(P) < d$ unless P contains S_d. So it is natural to consider the following question:

Question 4.5.1. *Let P be a poset. Given that $|P|$ is a large and $\dim(P)$ is close to $|P|/2$, is it true that P must contain a standard example S_d with d close to $|P|/2$?*

On the one hand, the results of EKT warn us that the dimension will have to be *very* close to $|P|/2$ in order to have such a result. This follows from the observation that when $p = 1/2$, almost surely, $\dim(P) > n - cn/\log n$ while the largest standard example S_d contained in P is almost surely of size $d \leq 2 \log n$ (as the cover graph of S_d contains a complete bipartite graph $K_{d/2,d/2}$). Nevertheless, an affirmative answer to this question was given by Biró, Hamburger, Pór and Trotter in [5].

Theorem 4.5.2. *For every positive integer c, there is an integer $f(c) = O(c^2)$ so that if $n > 10f(c)$ and P is a poset with $|P| \leq 2n + 1$ and $\dim(P) \geq n - c$, then P contains a standard example S_d with $d \geq n - f(c)$.*

From below, it is shown in [5] that $f(c) = \Omega(c^{4/3})$. However, if one examines the proof of Theorem 4.5.2 as presented in [5], there does not appear to be any way the argument could be tweaked to lower the exponent on c below 2.

So in an effort to settle the question of the correct exponent on c in the function $f(c)$, it is natural to turn to the study of random posets—and just to get the general idea, we temporarily ignore $\log n$ factors. Now we consider random posets from $\Omega(n, p)$ with $p = 1 - 1/\sqrt{n}$. So on average, each point in $\min(P)$ is incomparable to $\sqrt{n}$ points in $\max(P)$, and vice versa. An easy calculation shows that the the largest standard example in P would have size $\sqrt{n}$. Furthermore, the upper bound on dimension given in Theorem 4.1.6 is $n - \sqrt{n}$. If this bound is correct from below, then it follows that the exponent 2 is correct.

But to our dismay, the lower bound from EKT in Theorem 4.1.3 was not strong enough and it only implies: $\dim(P) > n - n/\log n$. This is not nearly enough to say anything of interest about the about the extremal problem (in fact, it only shows that $f(c) = \Omega(c \log c)$). So this shortcoming was another reason for revisiting the subject of dimension for random posets.

Our improved results do allow us to say something about the function $f(c)$. In the following discussion, we simply ignore multiplicative constants. Setting $q = n^{-1/4}$, we know from Theorem 4.1.11 that almost surely, $\dim(P) \geq n - n^{3/4}/\log^{(2-\epsilon)/3} n$. Furthermore, an easy calculation shows that the largest standard example S_d in P satisfies, almost surely, $d \leq n^{3/4} \log n$. Combining these two observations, we have $f(c) = \Omega(c^{4/3} \log^{8/9} c)$.

But just using the bounds obtained thus far, there does not seem to be any way to push the exponent past 4/3. However, in [5], it is shown that $f(c) = \Omega(c^{4/3})$, using a constructive approach, and the details of this approach will prove important here.

In [5], the authors consider a finite projective plane $\mathbb{P}$ with $m^2 + m + 1$ points and $m^2 + m + 1$ lines. Setting $n = m^2 + m + 1$, they associate with $\mathbb{P}$ a bipartite poset $P = A \cup A'$ defined in the following natural way. A is taken to be the points of $\mathbb{P}$ while A' is the set of lines in $\mathbb{P}$. Then $a < a'$ in P if and only if the point a is not incident with the line a' in $\mathbb{P}$. Note that in finite projective plane $\mathbb{P}$ any two distinct points determine a line, therefore we have the following observation.

Observation. In the finite projective plane $\mathbb{P}$, two points determine a unique line, so the poset P is 2-mixed.

The following result is implicit in the analysis of the dimension of this bipartite poset as given in [5].

Proposition 4.5.3. *Let $P = A \cup A'$ be a 2-mixed bipartite poset with* $\mathrm{Inc}(A, A') \neq \emptyset$*. Then* $\mathrm{Idim}(P)$ *is the minimum size of a maximal antichain matching in P.*

It is known (see [5] for details) that the largest standard example in P has size at most $1 + m^{3/2}$ while the minimum size of a maximal matching is $m^2 + m + 1 - m^{3/2}$. These observations show $\dim(P) \geq m^2 + m + 1 - m^{3/2}$ and $f(c) = \Omega(c^{4/3})$.

For general posets, these observations suggest that we should have a complementary notion of mixing. We say that a bipartite poset $P = A \cup A'$ is r-blocked when there does not exist a pair (V, V'), where V and V' are r-element subsets of A and A' respectively, with $v < v'$ in P for every $(v, v') \in V \times V'$.

With this notion in place, we can now express the following useful lemma. Again, it is implicit in [5].

Lemma 4.5.4. *If P is a 2-mixed, r-blocked bipartite poset in $\Omega(n, p)$, then* $\mathrm{Idim}(P) > n - r$*.*

Proof. Suppose $\mathrm{Idim}(P) = d$ and let $\mathcal{F} = \{L_1, L_2, \ldots, L_d\}$ be a realizer of P in the standard form guaranteed by Lemma 4.2.4. Define the subsets T, T', $M = A - M$ and $M' = A' - T'$ as has been our standard practice. Then we claim that $x < y$ in P for every $(x, y) \in M \times M'$. To see this, suppose to the contrary that there is some pair $(x, y) \in M \times M'$ with $x \parallel y$ in P. Then there is some j with $x > y$ in L_j. However, this implies that both elements of $\{a_j, x\}$ are incomparable with both elements of $\{a'_j, y\}$ in P, which would imply that P is not 2-mixed.

But since P is r-blocked, we know that $n - d < r$. □

4.5.1 An Application of the Lovász Local Lemma

With Lemma 4.5.4 in mind, we can make progress on our extremal problem using the asymmetric form of the celebrated Lovász Local Lemma [20]. The technical details of our application of the lemma follow along lines which are quite similar to the discussion on page 72 in the well known text [2] by Alon and Spencer.

Let n be a large integer and consider parameters q and a reasonably large integer r. Both q and r should be taken as variables whose values are to be determined.

We then consider a family $\mathcal{E} \cup \mathcal{F}$ of events. Events in $\mathcal{E}$ are called Type 1 events, while events in $\mathcal{F}$ are called Type 2 events. Furthermore, $|\mathcal{E}| = \binom{n}{2}^2$ and $|\mathcal{F}| = \binom{n}{r}^2$.

For each pair (U, U') where U is a 2-element subset of A and U' is a 2-element subset of A', a Type 1 event $E(U, U')$ occurs when $u \parallel u'$ in P for all $(u, u') \in U \times U'$. Then we have $\Pr(E(U, U')) = q^4$.

For each pair (V, V') where V is an r-element subset of A and V' is an r-element subset of A', a Type 2 event $F(V, V')$ occurs when $v < v'$ in P for all $(v, v') \in V \times V'$. Then we have $\Pr(F(V, V')) = (1-q)^{r^2} = e^{-qr^2}$.

Using generous estimates on the sizes of neighborhoods, it follows from the Local Lemma that if we can find a $0 < q < 1$, an $0 < r < n$ and two real numbers $0 \leq x < 1$ and $0 \leq y < 1$ such that

$$q^4 \leq x(1-x)^{4n^2}(1-y)^{n^{2r}}$$

and

$$(1-q)^{r^2} \leq y(1-x)^{r^2 n^2}(1-y)^{n^{2r}}$$

then with positive probability that a bipartite poset P in $\Omega(n, p)$ is 2-mixed and r-blocked.

Our goal is to determine the largest possible value of r for which there are choices for the remaining parameters which satisfy these constraints. We start by setting $(1-y)^{n^{2r}} = e^{-1}$, i.e., we want $yn^{2r} = 1$. Also we set $x = 9q^4$. Then we will keep $1/x \geq 4n^2$ so that the term $(1-x)^{4n^2}$ can be approximated by e^{-1}. Therefore the first constraint will always be satisfied and we can focus on the second.

Now the key idea is to set

$$y = (1-x)^{r^2 n^2} = e^{-qr^2/3}.$$

In that way the second inequality will also hold.

Plug in $x = 9q^4$, we get

$$(1 - 9q^4)^{r^2 n^2} = e^{-9q^4 r^2 n^2} = e^{-qr^2/3}.$$

Therefore we have $q = 1/(3n^{2/3})$.

For variable y, we have

$$e^{qr^2/3} = y^{-1} = n^{2r} = e^{2r \log n}.$$

Hence $r = 6 \log n / q = 18n^{2/3} \log n$.

We conclude that there exists a bipartite poset $P = A \cup A'$ which is 2-mixed and r-blocked. It follows from Lemma 4.5.4 that $\mathrm{Idim}(P) > n - r$. On the other hand, the largest value of d for which P contains a standard example S_d must clearly satisfy $d < 2r$. Setting $c = r = 18n^{2/3} \log n$, we see that $f(c) = \Omega\big(c^{3/2} / \log^{3/2} c\big)$.

4.5.2 The Implications of Being 2-Mixed

In the preceding section, Lemma 4.5.4 was applied to an exceptionally rare poset. Now we use the Lemma 4.5.4 to complete the proofs of Theorems 4.1.12 and 4.1.13, starting with the lower bound.

Suppose that $\log^{1+\epsilon} n/n < q \le n^{-4/5} \log^{1/5} n$. Let Y be a random variable which counts pairs (U, U') where U and U' are 2-element subsets of A and A' respectively with $u \parallel u'$ in P for all $(u, u') \in U \times U'$. Then we have $\mathrm{E}[Y] = \binom{n}{2}^2 q^4$.

Lemma 4.5.5. *If $\log^{1+\epsilon} n/n < q \le n^{-4/5} \log^{1/5} n$, then almost surely $Y \le n^4 q^4$.*

Proof. We use Chebyshev's inequality to prove the lemma. Let $Y = X_1 + X_2 + \cdots + X_r$ where $r = \binom{n}{2}^2$ and each X_i is a indicator random variable, i.e., $X_i = 1$ if the corresponding 4-element set is in Y; $X_i = 0$ otherwise. We have

$$\begin{aligned}
\mathrm{Var}[Y] &= \sum_{i=1}^{r} \mathrm{Var}[X_i] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j] \\
&\leq \sum_{i=1}^{r} \mathrm{E}[X_i] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j] \\
&= \mathrm{E}[Y] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j] \\
&\leq \binom{n}{2}^2 q^4 + \binom{n}{2}^2 q^4 \cdot 4n^2 q^3 + \binom{n}{2}^2 q^4 \cdot 4nq^2 \\
&= \mathcal{O}(n^4 q^4)
\end{aligned}$$

By Chebyshev's inequality, we have

$$\begin{aligned}
\Pr(Y > n^4 q^4) &\leq \Pr(|Y - \mathrm{E}[Y]| > n^4 q^4/2) \\
&\leq 4\mathrm{Var}[Y]/n^8 q^8 \\
&= \mathcal{O}(n^{-4} q^{-4}) \\
&\leq C/\log^{4+4\epsilon} n
\end{aligned}$$

Therefore almost surely we have $Y \leq n^4 q^4$. □

Now we are ready to prove Theorem 4.1.13 and 4.1.12.

Proof of Theorem 4.1.13. By Lemma 4.4.1, almost all posets in P are $3\log n/q$-blocked. Let P be a poset which is $3\log n/q$ blocked and has $Y \leq n^4 q^4$. Then let $Q = B \cup B'$ be a bipartite subposet of P with $|B| = |B'| = n - n^4 q^4$ so that Q is 2-mixed. Also, since P is almost surely $3\log n/q$-blocked, so is Q. This implies, by Lemma 4.5.4, that almost surely $\mathrm{Idim}(Q) > n - n^4 q^4 - 3\log n/q$. Hence almost surely $\mathrm{Idim}(P) \geq \mathrm{Idim}(Q) > n - n^4 q^4 - 3\log n/q$. But in view of our bound on q, we can conclude that $\mathrm{Idim}(P) > n - 4\log n/q$. This completes the proof of Theorem 4.1.13. □

Proof of Theorem 4.1.12. For the upper bound, we essentially reverse the above argument. We let Z_r be the random variable which counts the number of pairs (V, V')

for which $|V| = |V'| = r$, $v < v'$ in P for all $(v, v') \in V \times V'$, and there is an antichain matching in the bipartite subposet $Q = (A - V, A' - V')$. Again applying standard probabilistic techniques, almost surely $Z_r \geq 1$ when $r \leq \log n/q$. However, this shows that $\mathrm{Idim}(P) \leq n - \log n/q$. This completes the proof of Theorem 4.1.12. $\square$

Here it is important to note that for the first time, we have captured the "rebounding" property for $\mathrm{Idim}(P)$ in the upper bounds. Also, note that the upper bound in this theorem holds for an even broader range of values of q than what is included in the hypothesis. However, when $q \leq n^{-1/2}$, the upper bound from Theorem 4.1.7 is better.

4.6 A Second Extremal Problem

With the machinery we have developed thus far, we can also say something interesting about another extremal problem involving conditions which force a poset to contain a large standard example—although now we will be concerned with random posets where p is very small.

For integers d and n with $n \geq 2d+1$, define $f(n, d)$ to be the least positive integer so that if P is a poset with $|P| = n$ and $\dim(P) \geq f(n, d)$, then P contains S_d. Our primary interest now is in determining the behavior of $f(n, d)$ when d is fixed and n tends to infinity. This problem was first posed in [41], and then referenced on page 274 in EKT. Here we obtain a better result, and we remove the requirement that d be large.

For historical reasons, the value of $f(n, 2)$ has been studied—albeit with different notation and terminology—for many years. No doubt this results from the fact that the class of posets excluding the standard example S_2 constitute the class of *interval orders*. Combining results of several authors (see the discussion in [5]), the value of $f(n, 2)$ is known to within a small additive constant. However, as a crude estimate, we have $f(n, 2) = (1 + o(1)) \lg \lg n$.

Here, we will prove the following result.

Theorem 4.6.1. *For all $d \geq 3$,*

$$f(n,d) \geq \frac{n^{1-\frac{2d-1}{d(d-1)}}}{9\log n}.$$

Proof. We fix a value of $d \geq 3$ and then take p to have the following value

$$p = n^{-\frac{(2d-1)}{d(d-1)}}. \tag{4}$$

We then consider three events E_1, E_2 and E_3, and show that there exists a poset $P \in \Omega(n,p)$ for which all three events occur. The event E_1 is simply that P have a complete antichain matching. Almost all posets satisfy this property, so we can comfortably take $\Pr(\neg E_1) < 1/6$.

Fix the value of s at $s = 3\log n/p$. In view of Lemma 4.4.1, almost surely a poset P in $\Omega(n,p)$ is s-mixed. Event E_2 occurs when P is s-mixed. Again, being generous, we can take $\Pr(\neg E_2) < 1/6$.

We need the following technical lemma.

Lemma 4.6.2. *If P is a poset in $\Omega(n,p)$ and P is s-mixed, then P does not contain a bipartite subposet $Q = V \cup V'$ with $|V| = |V'| = 2s$ satisfying the following condition: There is a a complete antichain matching $\mathcal{M}$ in Q, as evidenced by the labellings $V = \{v_1, v_2, \ldots, v_{2s}\}$ and $V' = \{v'_1, v'_2, \ldots, v'_{2s}\}$ for which there is a linear extension L of Q with $v_i > v'_i$ in L for each $i = 1, 2, \ldots, 2s$.*

Proof. Suppose P is a poset for which such a bipartite subposet and the associated linear extension can be found. After a relabelling, we may assume that $v_1 > v_2 > v_3 > \cdots > v_{2s}$ in L. However, this implies that $v_i > v'_j$ in L whenever $1 \leq i \leq s$ and $s+1 \leq j \leq 2s$, which implies that P is not s-mixed. □

Next, we let X be the random variable that counts the number of copies of S_d in P. Then the expected value of X is given by

$$\mathrm{E}[X] = \binom{n}{d}^2 d!(1-p)^d p^{d(d-1)} < n/d! \leq n/6.$$

Let E_3 be the event that occurs when $X < n/3$. Then $\Pr(\neg E_3) < 1/2$.

Since $\Pr(\neg E_1) + \Pr(\neg E_2) + \Pr(\neg E_3) < 1/6 + 1/6 + 1/2 = 5/6$, there exists a poset P in $\Omega(n, p)$ for which all three events occur.

Let $\mathcal{M} = (A, A')$ be an antichain matching in P as evidenced by the labellings $A = \{a_1, a_2, \ldots, a_n\}$ and $A' = \{a'_1, a'_2, \ldots, a'_n\}$. Without loss of generality, we may assume that any copy of S_d contained in P (there are less than $n/3$ of them) contains at some point in $\{a_i : 2n/3 < i \leq n\} \cup \{a'_i : 2n/3 < i \leq n\}$. Hence there are *no* copies of S_d in the bipartite subposet $Q = B \cup B'$ where $B = \{a_i : 1 \leq i \leq 2n/3\}$ and $B' = \{a'_i : 1 \leq i \leq 2n/3\}$.

Since Q is a subposet of P, we know that $\text{Idim}(P) \geq \text{Idim}(Q)$. Now let $\mathcal{F}$ be any family of linear extensions of Q which is a realizer of $B \times B'$. Then $\mathcal{F}$ must reverse the pairs in $\{(a_i, a'_i) : 1 \leq i \leq 2n/3\}$. However, in view of Lemma 4.6.2, we conclude that

$$\dim(P) \geq \text{Idim}(P) \geq \text{Idim}(Q) \geq \frac{2n/3}{2s} = \frac{pn}{9 \log n} = \frac{n^{1-\frac{2d-1}{d(d-1)}}}{9 \log n}.$$

This completes the proof of Theorem 4.6.1. □

Upper bounds on $f(n, d)$ when $d \geq 3$ are more challenging. Indeed the following result of Biró, Hamburger and Pór [4] is all that we know.

Theorem 4.6.3. *For every $d \geq 3$ and every $\epsilon > 0$, there is an integer n_0 so that if $n > n_0$ and P is a poset with $|P| = 2n$ and P does not contain the standard example S_d, then* $\dim(P) < \epsilon n$.

One may wonder why we did not use the EKT result on the expected value of $\text{Idim}(P)$. The problem is that in order to get the smallest possible value on the exponent in the expression for p, we need to have the order of the expected number of copies of S_d behave like n and not like $\text{Idim}(P)$. This means that we will remove a very large number of points from P and while it might be natural to believe that we can do this in a way that the subposet Q which does not contain any copies of S_d

has very large dimension, it would require knowledge of the concentration about the expected value of dimension, and we are a long way away from understanding this issue.

On the other hand, the technique used in this proof provides a very simple argument that, for example when $p = n^{-1/2}$, almost surely $\mathrm{Idim}(P) = \Omega(n^{1/2}/\log n)$. So the EKT machinery only manages to move the $\log n$ term to the numerator in this lower bound. This detail is essential for understanding the accuracy of the Füredi-Kahn bounds, but the elementary approach at least gives the right exponent.

4.7 Some Comments on Open Problems

The original EKT paper contained a list of 12 problems. Here we have made substantive progress on one of them (Problem 5.3) and solved Problem 5.6 completely by answering the question posed in the negative. We have made substantive progress on Problem 5.8, although much work remains to be done. In particular, we should now ask for the true behavior of the expected value of $\dim(P)$ when $n^{-4/5} < q \leq n^{-1/4}$, the interval where we are unable to determine the expected value of $n - \dim(P)$ up to a multiplicative constant.

Of course, the two related extremal problems discussed in the preceding two sections are still a challenge. We are hesitant to offer a conjecture on the correct exponent of the function $f(c)$, as there are too many examples where the Lovász Local Lemma provides useful information but not the entire answer.

For the second extremal problem, it would be very interesting to show that for each $d \geq 3$, there is a constant $c_d > 0$, so that $f(n, d) < n^{1-c_d}$, although it is not clear that such a constant exists, even when $d = 3$.

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Combinatorial Problems for Graphs and Partially Ordered Sets

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99 Pages

Directed by Professor William T. Trotter

This dissertation has Three principal components. The first component is about the dimension of posets with matchings in comparability and incomparability graphs. In 1951, Hiraguchi proved that for any finite poset P, the dimension of P is at most half the size of P. We develop some new inequalities for the dimension of finite posets. These inequalities are then used to bound dimension in terms of the maximum size of matchings. We prove that if the dimension of P is d and d is at least 3, then there is a matching of size d in the comparability graph of P, and a matching of size d in the incomparability graph of P. The bounds in above theorems are best possible, and either result has Hiraguchi's theorem as an immediate corollary.

In the second component, we focus on an extremal graph theory problem whose solution relied on the construction of a special poset. In 1959, Paul Erdős, in a landmark paper, proved the existence of graphs with arbitrarily large girth and large chromatic number using probabilistic method. In a 1991 paper of Kříž and Nešetřil, they introduced a new graph parameter eye(G). They show that there are graphs with large girth and large chromatic number among the class of graphs having eye parameter at most three. Answering a question of Kříž and Nešetřil, we were able to strengthen their results and show that there are graphs with large girth and large chromatic number among the class of graphs having eye parameter at most two.

The last component is about random poset—the poset version of the Erdős–Rényi random graph. In 1991, Erdős, Kierstead and Trotter (EKT) investigated random height 2 posets and obtained several upper and lower bounds on the dimension of the random posets. Motivated by some extremal problems involving conditions which force a poset to contain a large standard example, we were compelled to revisit this

subject. Our sharpened analysis allows us to conclude that as p approaches 1, the expected value of dimension first increases and then decreases, a subtlety not identified in EKT. Along the way, we establish connections with classical topics in analysis as well as with latin rectangles. Also, using structural insights drawn from this research, we are able to make progress on the motivating extremal problem with an application of the asymmetric form of the Lovász Local Lemma.