1. Computability, Complexity and Algorithms

(a): \textbf{Count} $s - t$ \textbf{Paths in DAGs}: Let $G(V, E)$ be a directed acyclic graph given in adjacency list representation, and let $s \in V$ and $t \in V$ be distinct vertices. Give an $O(|V| + |E|)$ algorithm that computes the number of distinct paths from $s$ to $t$ in $G$.

(b): \textbf{Count} $s - t$ \textbf{Paths in General Directed Graphs}: Let $G(V, E)$ be a general directed graph given in adjacency matrix representation, and let $s \in V$ and $t \in V$ be distinct vertices. Argue that, if there is a polynomial-time algorithm that computes the number of distinct paths from $s$ to $t$ in $G$, then there is a polynomial-time algorithm that decides Hamiltonicity in general directed graphs.

2. Analysis of Algorithms

\textbf{Matrix Identity Testing}

- Recall the Schwartz-Zippel lemma:

\textbf{Lemma 1 (Schwartz-Zippel Lemma)} Let $p(x_1, \ldots, x_n)$ be a nonzero polynomial of $n$ variables with degree $d$. Let $S$ be a finite subset of $\mathbb{R}$, with at least $d$ elements in it. If we assign $x_1, \ldots, x_n$ values from $S$ independently and uniformly at random, then

$$\mathbb{P}[p(x_1, \ldots, x_n) = 0] \leq \frac{d}{|S|}.$$ 

Using the aforementioned lemma, design a randomized algorithm to test whether $AB = C$, where $A, B, C$ are three $n \times n$ matrices. Analyze the probability with which it will succeed, and analyze its runtime.

- Explain how to “boost” the above algorithm to succeed with probability $1 - \delta$.

3. Theory of Linear Inequalities

Let $e^k \in \mathbb{R}^n$ for $k = 0, \ldots, n - 1$ denote the vector with the first $k$ entries being 1 and the following $n - k$ entries being $-1$. Let $S = \{e^0, e^1, \ldots, e^{n-1}, -e^0, \ldots, -e^{n-1}\}$, i.e., $S$ consists of all vectors consisting of +1 followed by $-1$ or vice versa.

1. Consider any vector $a \in \{-1, 0, 1\}^n$ such that

(a) $\sum_{i=1}^{n} a_i = 1$, and

(b) for all $j = 1, \ldots, n - 1$, we have $0 \leq \sum_{i=1}^{j} a_i \leq 1$. 

Show that $\sum_{i=1}^{n} a_i x_i \leq 1$ and $\sum_{i=1}^{n} a_i x_i \geq -1$ are valid inequalities for $\text{conv}(S)$.

2. Show that any such inequality defines a facet of $\text{conv}(S)$.

4. Combinatorial Optimization

Assume $n$ is odd, and $G = (V, E)$ is a graph with $|V| = n$, $|E| = 2n - 2$, such that $G$ is the union of two edge-disjoint spanning trees. Assume furthermore that half of the edges are colored red, the other half blue (where the coloring of edges is unrelated to the spanning trees). Show that $G$ contains a spanning tree where exactly half of the edges are red and half of them blue.

5. Graph Theory

Let $d$ be a positive integer and let $G$ be a graph with average degree at least $8d$. Show that $G$ contains a $d$-connected subgraph whose edges can be oriented so that the resulting digraph has no directed path on three vertices.

6. Probabilistic methods

Let $S_n$ be a random string of length $n$, where each character is, independently, chosen uniformly at random from the alphabet $\mathcal{A} := \{A, \ldots, Z\}$. For each $n$, let $H_n \in \mathcal{A}^m$ be a given string of length $m = m(n) \geq 0$. We say that $S_n$ contains $H_n$ if $S_n$ it contains a consecutive substring of length $m$ which equals $H_n$. Find a threshold function $m^* = m^*(n)$ such that

$$\Pr(S_n \text{ contains } H_n) \rightarrow \begin{cases} 1 & m = o(m^*) , \\ 0 & m = \omega(m^*) . \end{cases}$$

7. Algebra

Let $p$ be a prime number. Show that if $G$ is a finite $p$-group, and if $N \trianglelefteq G$ is a normal subgroup of order $p$, then $N$ is contained in the center of $G$.

7. Linear Algebra

Let $A$ be a bistochastic matrix, that is a real $n \times n$ matrix such that

$$A_{i,j} \geq 0 \quad \forall i, j \quad \sum_{i=1}^{n} A_{i,j} = 1 \quad \forall j \quad \sum_{j=1}^{n} A_{i,j} = 1 \quad \forall i .$$
Let $a = \min_{i,j} A_{i,j}$ and let $v \in \mathbb{R}^n$ be such that $\sum_{i=1}^n v_i = 0$.

(a) Show that 
\[ \|Av\|_1 \leq (1 - na)\|v\|_1, \]
where $\|v\|_1 = \sum_{i=1}^n |v_i|$. Is the estimate sharp? That is, can you find $A$ and $v$ as above such that 
\[ \|Av\|_1 = (1 - na)\|v\|_1? \]

(b) Show that 
\[ \|Av\|_\infty \leq (1 - na)\|v\|_\infty, \]
where $\|v\|_\infty = \max_i |v_i|$. Is the estimate sharp? That is, can you find $A$ and $v$ as above such that 
\[ \|Av\|_\infty = (1 - na)\|v\|_\infty? \]