1. Graph Theory

Let G be a simple graph with maximum degree d. Prove that $E(G)$ can be decomposed into pairwise disjoint (possibly empty) matchings $M_1, \ldots, M_{d+1}$ such that $-1 \leq |M_i| - |M_j| \leq 1$ for all $1 \leq i, j \leq d+1$.

Solution: By Vizing's theorem, G has a $d+1$ edge-coloring. Hence, $E(G)$ can be partitioned into $d+1$ (possibly empty) matchings, say $M_1, \ldots, M_{d+1}$. Let $a = \max\{|M_i| : i = 1, \ldots, d+1\}$ and $b = \min\{|M_i| : i = 1, \ldots, d+1\}$. Choose $M_1, \ldots, M_{d+1}$ such that

(1) $a - b$ is minimal, and

(2) subject to (1), the number of pairs (M_i, M_j) with $|M_i| - |M_j| = a - b$ is minimal.

If $a - b \leq 1$, we are done. So we may assume that there is a pair (M_i, M_j) with $|M_i| - |M_j| = a - b \geq 2$. We may assume $i = 1$ and $j = 2$, by an appropriate relabeling (if necessary). Consider $H := G[M_1 \cup M_2]$, the subgraph of G induced by $M_1 \cup M_2$. Note that the components of H are paths and/or cycles whose edges alternate in M_1 and M_2.

Since $|M_1| - |M_2| \geq 2$, there is a component of H, say P, that is a path with one more edge in M_1 than in M_2. Let $M_1' := M_1 \Delta P$ and $M_2' := M_2 \Delta P$, where Δ denotes the symmetric difference. Then $M_1' \cup M_2' = M_1 \cup M_2$, $1 \leq |M_1'| - |M_2'| < |M_1| - |M_2|$, and M_1' and M_2' are disjoint matchings.

Let $M_k' := M_k$ for all $k = 3, \ldots, d+1$. It is straightforward to verify that the partition $M_1', \ldots, M_{d+1}'$ violates (1) or (2).

2. Probability

Assume that a Markov text $X_0, X_1, X_2, X_3, \ldots, X_n$ with letters from the alphabet $S = \{a, b, c\}$ has its transition probabilities given in the following transition matrix:

$$\begin{pmatrix} p_{a\to a} & p_{a\to b} & p_{a\to c} \\ p_{b\to a} & p_{b\to b} & p_{b\to c} \\ p_{c\to a} & p_{c\to b} & p_{c\to c} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.8 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

a) What are the stationary probabilities $\pi(a), \pi(b), \pi(c)$ equal to? (i.e. what are the long term frequencies of the a's, b's and c's in the text?)
b) Given that we start in state a, what is the expected time until we visit state b for the first time?

Solution: (a) The solution is obtained by solving the vector equation.

$$\vec{\pi} = \vec{\pi} P \tag{1}$$

where $\vec{\pi} = (\pi_a, \pi_b, \pi_c)$ and P is the transition matrix. Solving the vector equation 1, we obtain

$$\vec{\pi} = (0.25, 0.5, 0.25).$$

This could also be obtained by a simple symmetry argument.
(b) Let $T = \min\{t \geq 0 | X_t = b\}$. We want to find

$$E[T|X_0 = a] = E_a[T].$$

For this we condition on the first step and find

$$E_a[T] = E_a[T|X_1 = a]P_a(X_1 = a) + E_a[T|X_1 = b]P_a(X_1 = b) + E_a[T|X_1 = c]P_a(X_1 = c)$$

where $E_a[T|X_1 = b] = 1$, $P_a(X_1 = b) = p_{a\to b}$, $P_a(X_1 = c) = p_{a\to c}$ and $E_a[T|X_1 = c] = 1 + E_c[T]$. Hence, we find the equation

$$E_a[T] = p_{a\to a}(1 + E_a[T]) + p_{a\to b} + (1 + E_c[T])p_{a\to c}.$$

Similarly, one gets

$$E_c[T] = (1 + E_a[T])p_{c\to a} + p_{c\to b} + p_{c\to c}(1 + E_c[T]).$$

The last two equation above form a system of equations which can be solved. We know the transition probabilities and we put $x = E_a[T]$ and $y = E_b[T]$ as unknowns. In this manner we obtain

$$x = 0.5(1 + x) + 0.2 + (1 + y)0.3$$

and

$$y = 0.3(1 + x) + 0.2 + (1 + y)0.5$$

which can be solved. The solution is $x = y = 5$. (One could also see that given that we start in a or c the time until we visit b is like a geometric variable with parameter 0.2).

3. Analysis of Algorithms

Given an undirected graph $G = (V, E)$, let C be a coloring of G where each vertex is colored either red, green, or blue. We say that an edge is monochromatic in C if the colors on its endpoints agree, and bichromatic if they disagree. An *ideal coloring* is any coloring that maximizes the number of bichromatic edges. Unfortunately finding an ideal coloring is NP-hard.

(a) Let $M(G)$ be the number of monochromatic edges in an ideal coloring. Show that it is NP-hard to approximate $M(G)$ to within a factor of 10^{100}.

(b) Let $B(G)$ be the number of bichromatic edges in an ideal coloring. Give a randomized algorithm that outputs a coloring such that the expected number of bichromatic edges is at least $\frac{2}{3}B(G)$.

Solution: **(a)** If a graph is 3-colorable, then $M(G)$ is 0 and if it is not 3-colorable then $M(G) \geq 1$. Any approximation algorithm that estimates $M(G)$ within a constant factor would output $x > 0$ if the graph is 3-colorable and $x = 0$ if it is not, thus deciding whether the graph is 3-colorable. Therefore we can conclude that approximating $M(G)$ within any factor is NP-hard.

(b) Assign each vertex a random color from the set {Red, Blue, Green}. Each edge has a 2/3 chance of being bichromatic, so the expected number of bichromatic edges is $\frac{2}{3}|E| \geq \frac{2}{3}B(G)$.

4. Combinatorial Optimization

Let A be an integral $m \times n$ matrix and let b and c be integral m-dimensional vectors. Show that there exists an integral vector x with $Ax \in \{b, c\}$ if and only if there does not exist a vector y such that $y^T A$ is integral, $y^T b$ is not an integer, and $y^T c$ is not an integer.

Solution. A solution is available upon request.

5. Theory of Linear Inequalities

Let $P = \{x : Ax \leq b,\ \mathbf{0} \leq x \leq \mathbf{1}\}$ be a rational polytope of dimension d, where A is an $m \times n$ matrix, b is an m-dimensional vector, and $\mathbf{0}$ and $\mathbf{1}$ represent n-dimensional vectors with all components 0 and 1, respectively. Suppose P does not contain any integer vectors. Show that the Chvátal rank of P is no greater than d.

Solution. A solution is available upon request.

6. Algebra

Let G be a finite group acting on a set X, and let H be a normal subgroup of G. If $x \in X$, show that the G-orbit of x is a union of at most $[G : H]$ H-orbits of X, each having the same cardinality.

Solution: Replacing X by the orbit of x, we may assume that G acts transitively on X. Let $H \cdot x_1, \ldots, H \cdot x_k$ be the distinct H-orbits. For each i, choose $g_i \in G$ such that $g_i \cdot x_1 = x_i$. We claim that:

(a) If $i \neq j$ then the left cosets $g_i H$ and $g_j H$ are different.

(b) The action of g_i induces a bijection $H \cdot x_1 \to H \cdot x_i$.

Claim (a) implies that the number of H-orbits is at most $|G/H|$, and Claim (b) implies that each H-orbit has the same size. So it remains to prove the two claims.

For (a), first note that since H is normal in G, left cosets are the same as right cosets. If $g_i H = g_j H$ with $i \neq j$, then $hg_i = g_j$ for some $h \in H$. Thus $(hg_i) \cdot x_1 = h \cdot x_i = g_j \cdot x_1 = x_j$. Therefore $x_j = h \cdot x_i$, contradicting our assumption that x_i and x_j live in different H-orbits.

For (b), note that if $h \cdot x_1 \in H \cdot x_1$, then

$$\begin{aligned} g_i \cdot (h \cdot x_1) &= (g_i h) \cdot x_1 \\ &= (h' g_i) \cdot x_1 \qquad \text{for some } h' \in H \\ &= h' \cdot x_i \in H \cdot x_i, \end{aligned}$$

so the action of g_i induces a map $\psi : H \cdot x_1 \to H \cdot x_i$. The action of g_i^{-1} gives an inverse to this map, so ψ must be a bijection.

7. Randomized Algorithms

Consider the following scheme for shuffling a deck of n cards labelled $c_1, c_2, \ldots, c_n$. For $i = 1, \ldots, n$, let $X_t(i)$ denote the card in the i-th position at time t. Let X_0 be an arbitrary ordering of the cards. For $t \geq 1$, given X_{t-1} define X_t as follows:

- Choose position i uniformly at random from $\{1, \ldots, n\}$ and card c_j uniformly at random from $\{c_1, \ldots, c_n\}$.
- Swap the card in position i with card c_j. I.e., let $X_{t+1}(i) = c_j$ and let $X_{t+1}(k) = X_t(i)$ where $k = X_t^{-1}(c_j)$ is the position of card c_j in X_t.
- For $\ell \notin \{i, k\}$, let $X_{t+1}(\ell) = X_t(\ell)$.

Show a coupling argument to upper bound the mixing time of this Markov chain, within a constant factor of optimal is fine. Recall, the mixing time is defined to be the number of steps (from the worst initial state) to get within variation distance $\leq 1/4$ of the uniform distribution.

Solution. Consider an arbitrary pair X_0, Y_0. Couple the evolution of this pair of processes using the identity coupling so that in every time step both chains choose the same position i and card c_j for the update.

Let D_t denote the number of positions where the chains X_t and Y_t differ. Note, after the coupled move we have $X_{t+1}(i) = Y_{t+1} = c_j$. Thus, if we choose a position i where they differ (i.e., $X_t(i) \neq Y_t(i)$) and a card c_j which is in different positions in the two chains then after the coupled move we have $D_{t+1} \leq D_t - 1$. In the other 3 scenarios (depending on whether $X_t(i) = Y_t(i)$ and/or card c_j is in the same position in X_t and Y_t), it is easy to check that $D_{t+1} \leq D_t$. Therefore,

$$\mathbf{Pr}(D_{t+1} \leq D_t - 1) \geq (D_t/n)^2$$

Hence the expected time t until $D_t = 0$ is $O(n^2)$. By Markov's inequality, it follows that $\mathbf{Pr}(X_t \neq Y_t) \leq Pr(D_t = 0) < 1/8$ for $t = O(n^2)$.

7. Approximation Algorithms

Consider the following modification to the metric uncapacitated facility location problem. Define the cost of connecting city j to facility i to be c_{ij}^2. The c_{ij}'s still satisfy the triangle inequality (but the new connection costs, of c_{ij}^2, do not). Show that factor 3 primal-dual algorithm (given below), which uses the usual LP-relaxation and dual for the facility location problem, achieves an approximation guarantee of factor 9 for this case.

Phase 1

Raise the dual variable α_j for each unconnected city j uniformly at unit rate, i.e., α_j will grow by 1 in unit time. When $\alpha_j = c_{ij}$ for some edge (i, j), the algorithm will declare this edge to be *tight*. Henceforth, dual variable β_{ij} will be raised uniformly, and it goes towards paying for facility i. Each edge (i, j) such that $\beta_{ij} > 0$ is declared *special*. Facility i is said to be *paid for* if $\sum_j \beta_{ij} = f_i$. If so, the algorithm declares this facility *temporarily open*. Furthermore, all unconnected cities having tight edges to this facility are declared *connected* and facility i is declared the *connecting witness* for each of these cities. In the future, as soon as an unconnected city j gets a tight edge to i, j will also be declared connected and i will be declared the connecting witness for j. When all cities are connected, the first phase terminates.

Phase 2

Let F_t denote the set of temporarily open facilities and T denote the subgraph of G consisting of all special edges. Let T^2 denote the graph that has edge (u, v) iff there is a path of length at most 2 between u and v in T, and let H be the subgraph of T^2 induced on F_t. Find any maximal independent set in H, say I.

All facilities in the set I are declared *open*. For city j, define $\mathcal{F}_j = \{i \in F_t \mid (i, j) \text{ is special}\}$. Since I is an independent set, at most one of the facilities in $\mathcal{F}_j$ is opened. If there is a facility $i \in \mathcal{F}_j$ that is opened, then set $\phi(j) = i$. Otherwise, consider tight edge (i', j) such that i' was the connecting witness for j. If $i' \in I$, again set $\phi(j) = i'$. In the remaining case that $i' \notin I$, let i be any neighbor of i' in graph H such that $i \in I$. Set $\phi(j) = i$.

Solution. At the end of Phase 1 the algorithm clearly produces an integral feasible primal solution and a feasible dual solution to the LP relaxation of the facility location problem, with costs c_{ij}. Throughout the analysis that follows, the variables a_j and b_{ij} refer to the ones developed in Phase 1.

We will establish the performance quarantee by showing that

$$\sum_{i \in I} f_i + \sum_{j \in C} c_{\phi j, j} \leq 9 \sum_{j \in C} \alpha_j \leq 9\text{OPT} \quad (*)$$

Let a_j be the dual variable for a city j. If city j is connected to facility $\phi(j) \in I$ during Phase 2, and facility $\phi(j)$ was also a connecting witness for city j at the end of Phase 1, then city j is called *directly connected* to $\phi(j)$. Let C_D denote the set cities directly connected to facilities. If city j is connected to facility $\phi(j) \in I$ during Phase 2, but facility $\phi(j)$ was not a connecting witness for city j at the end of Phase 1, then city j is called *indirectly connected* to $\phi(j)$. Let $\bar{C_D}$ be the set of cities indirectly connected to facilities.

Fact 1: If city $j \in C_D$, then, for all facilities $i \in I$ such that j was temporarily connected to i during Phase 1, $a_{ij} = c_{ij} + b_{ij}$.

Proof: By the definition of temporary connectivity in Phase 1.

Fact 2: If city $j \in \bar{C_D}$, then, for all facilities $i \in I$, $b_{ij} = 0$.
Proof: By the definition of indirect connectivity.

Fact 3: For all open facilities $i \in I$ $f_i = \sum_{i \in C_D} b_{ij}$.
Proof: By the structure imposed during Phase 1, and by Facts 1 and 2 above.

Fact 4: If city $j \in \bar{C_D}$, and $\phi(j) = i \in I$, then, there is a city i' and a facility j', such that $\sqrt{c_{ij}} \leq \sqrt{c_{i'j}} + \sqrt{c_{i'j'}} + \sqrt{c_{ij'}}$ and $\max\{c_{i'j}, c_{i'j'}, c_{ij'}\} \leq a_j$.
Proof: Let i' be a connecting witness for city j. Since j is indirectly connected to i, (i, i') must be an edge in H. In turn, there must be a city, say j', such that (i, j') and (i', j') are both special edges. Let t_1 and t_2 be the times at which i and i' were declared temporarily open during Phase 1. Since edge (i', j) is tight,

$$c_{i'j} \leq \alpha_j.$$

We will also show that

$$c_{ij'} \leq \alpha_j \quad and \quad c_{i'j'} \leq a_j.$$

The rest of the fact follows by triangular inequality. Since edges $(i', j\prime)$ and $(i, j\prime)$ are tight, $\alpha_{j'} \geq c_{ij'}$ and $\alpha_{j'} \geq c_{i'j'}$. Since both these edges are special, they must both have gone tight before either i or i' is declared temporarily open. Consider the time $\min(t_1, t_2)$. Clearly, $\alpha_{j'}$ cannot be growing beyond this time. Therefore, $\alpha_{j'} \leq \min(t_1, t_2)$. Finally, since i' is the connecting witness for j, $\alpha_j \geq t_2$. Therefore, $\alpha_j \geq \alpha_{j'}$, and the required inequalities follow.

Fact 5: If city $j \in \bar{C_D}$ and $phi(j) = i$, then, $c_{\phi(j),j} \leq 9a_j$.
Proof: By Fact 4,

$$\begin{aligned} c_{\phi(j),j} &= \left(\sqrt{c_{i'j}} + \sqrt{c_{i'j'}} + \sqrt{c_{ij'}}\right) 2 \\ &\leq (3\sqrt{a_j})2 \\ &= 9a_j \end{aligned}$$

We are now ready to establish (*):

$$\begin{aligned} \sum_{i \in I} f_i + \sum_{j \in C} c_{\phi j,j} &= \sum_{j \in C_D} b_{ij} + \sum_{j \in C} c_{\phi j,j} \quad by\ Fact\ 3 \\ &= \sum_{j \in C_D} b_{ij} + \sum_{j \in C_D} c_{\phi j,j} + \sum_{j \in \bar{C_D}} c_{\phi j,j} \\ &= \sum_{j \in C_D} a_j + \sum_{j \in \bar{C_D}} c_{\phi j,j} \quad by\ Fact\ 1 \\ &\leq \sum_{j \in C_D} a_j + 9 \sum_{j \in C_D} a_j \quad by\ Fact\ 5 \\ &\leq 9 \sum_{j \in C} a_j \leq 9\text{OPT} \end{aligned}$$