## 1. Computability, Complexity and Algorithms

**(a)** Let $G(V, E)$ be an undirected unweighted graph. Let $C \subseteq V$ be a vertex cover of $G$. Argue that $V \setminus C$ is an independent set of $G$.
**(b)** Minimum cardinality vertex cover and maximum cardinality independent set are well known NP-complete problems. Suppose that you have a polynomial time approximation algorithm that, on input an undirected unweighted graph $G(V, E)$, outputs a vertex cover $C$ whose cardinality is at most 2OPT. Is the cardinality of the independent set $V \setminus C$ a constant factor approximation algorithm for maximum independent set? If yes give a proof, if no give a counter example.
**(c)** Give a polynomial time algorithm that, on input an undirected unweighted bipartite graph $G$, outputs a minimum cardinality vertex cover of $G$.

**Solution:** **(a)** $C$ is a vertex cover of $G$, therefore for every edge $\{u, v\} \in E$ it is the case that either $u \in C$, or $v \in C$, or both $u$ and $v$ belong to $C$. To argue that $I = V \setminus C$ is an independent set, we need to show that, for every pair of vertices $u' \in I$ and $v' \in I$, it is the case that $\{u', v'\} \notin E$. This is obviously true, since if $\{u', v'\} \in E$, then either $u' \in C$, or $v' \in C$, or both, contradicting the assumption that both $u'$ and $v'$ belong to $I = V \setminus C$.
**(b)** The anwer is no. Counter example. Suppose $G(V, E)$ is the complete bipartite graph, with $V = L \cup R$, $|L| = |R| = |V|/2$, and all edges having exactly one endpoint in $L$ and the other endpoint in $R$. Clearly $L$ is a minimum cardinality vertex cover and it has cardinality $|V|/2$, and $R$ is a maximum cardinality independent set and has cardinality $|V|/2$. Clearly, a factor 2 approximation algorithm for vertex cover may output $C = V = L \cup R$ which satisfies $|C| = |V| \leq 2\text{OPT}$. However, the complement of $C$ is $I = V \setminus C = \emptyset$, with $|I| = 0$, which is not a constant approximation for the cardinality of the minimum independent set $|R| = |V|/2$.
**(c)** Here is an algorithm:
Input: undirected, unweighted bipartite graph $G(V, E)$ with vertex bipartition classes $L$ and $R$.
Construct the standard flow network corresponding to $G$: $G' = (V', E'), s, t, c = 1 \forall e \in E$ .
$S$ is a mincut in $G'$, which can be found in polynomial time using standard maxfow.
Let $L_1 := L \cap S$, $L_2 := L \setminus S$, $R_1 := R \cap S$,$R_2 = R \setminus S$.
Let $B$ be the set of vertices in $R_2$ that have neighbors in $L_1$.
$C := L_2 \cup R_1 \cup B$.
output $C$.
<u>Fact</u> $C$ is a vertex cover of $G$.
<u>Proof.</u> The set $C$ covers all edges that have one endpoint in either $L_2$ or $R_1$, because $C$ includes all of $L_2$ and $R_1$. All remaining edges must have one endpoint in $L_1$ and the other endpoint in $R_2$. These edges are then clearly covered by $B$.
<u>Fact</u> $G$ has no vertex cover of cardinality smaller than $|C|$.
<u>Proof.</u> Let $k$ be the capacity of the cut $S$. Then $k = |L_2| + |R_1| + |edged(L_1, R_2)|$, consequently $k \geq |L_2| + |R_1| + |B| = |C|$. But $S$ is a mincut of $G'$, thus $k$ is equal to the mincut of $G'$, which is equal to the maxiflow in $G'$, which is equal to the size of the maximum cardinality matching of $G$. This means that $G$ has a matching of size $k$, and therefore every vertex cover of $G$ must have cardinality at least $k \geq |C|$.

## 2. Analysis of Algorithms

**1.** The *exact matching* problem is the following: Given a bipartite graph $G = (U, V, E)$ and an integer $k \leq n$, with $|U| = |V| = n$ and with a subset $E' \subset E$ of the edges colored red, an exact matching is a perfect matching with exactly $k$ red edges. Give randomized polynomial-time algorithms for:

(a) Testing if $G$ has an exact matching.

(b) If so find one.

**2.** Next consider an extension of this problem where two disjoint subsets $E_1$ and $E_2$ of edges are colored red and blue, respectively, and two integers $k_1, k_2$ are specified with $k_1 + k_2 \leq n$. Now we seek a perfect matching with $k_1$ red edges and $k_2$ blue edges. Repeat the two previous questions for this extended notion.

**Solution:** Let $A$ be the $n \times n$ adjacency matrix for $G$. Corresponding to each red edge $(i, j)$, replace $A(i, j)$ by the variable $x$. Let $A_x$ be the resulting matrix. Now $G$ has an exact matching iff the permanent of $A$ has the monomial $cx^k$, where $c > 0$. However, permanent is hard to compute. Instead, multiply each entry of $A_x$ by a randomly and independently picked number in the range $[0, 2n^2]$ to obtain the matrix $A'$, say.

Compute $|A'|$. By Schwartz's Lemma, if $G$ has an exact matching, then with high probability, the monomial with $x^k$ will have a non-zero coefficient and that will be proof of existence of exact matching. If $G$ has no exact matching, then the determinant will not have this monomial. If yes, the exact matching can be found using self-reducibility.

For the second part, in $A$, replace red edges by variable $x$ and blue edges by variable $y$ and again multiply each entry by a randomly and independently picked number in the range $[0, 2n^2]$ to obtain the matrix $A''$, say. Now with high probability in $|A''|$ the monomial $x^{k_1}y^{k_2}$ will have a non-zero coefficient iff $G$ has an exact matching. If yes, again it can be found via self-reducibility.

## 3. Theory of Linear Inequalities

For a system $Ax \leq b$ of $m$ rational linear inequalities and a set $S \subseteq \{1, \ldots, m\}$ let

$$A_S x = b_S,\ A_{\bar{S}} x \leq b_{\bar{S}} \tag{1}$$

denote the system obtained by setting each inequality in $S$ to equality while keeping each inequality in $\bar{S} = \{1, \ldots, m\} \setminus S$ as an inequality.

Suppose the system (1) has no solution for some specified set $S$. Then, by Farkas's Lemma, there exists a vector $(y_S, y_{\bar{S}})$ such that

$$y_S^T b_S + y_{\bar{S}}^T b_{\bar{S}} < 0,\ y_S^T A_S + y_{\bar{S}}^T A_{\bar{S}} = 0,\ y_{\bar{S}} \geq 0. \tag{2}$$

Due to the equality constraints, the vector $y_S$ might have negative components. Notice, however, that we may scale $(y_S, y_{\bar{S}})$ so that it satisfies

$$y_S^T b_S + y_{\bar{S}}^T b_{\bar{S}} < 0,\ y_S^T A_S + y_{\bar{S}}^T A_{\bar{S}} = 0,\ y_S \geq -1,\ y_{\bar{S}} \geq 0. \tag{3}$$

Now (2) necessarily has an integral solution (since it has a rational solution), but (3) may not be solvable in integers. We say that the infeasibility of (1) can be *proven integrally* if (3) does in fact have an integral solution.

Prove the following theorem.

**Theorem 1** *Let $A$ be an integral matrix and let $b$ be a rational vector such that $Ax \leq b$ has at least one solution. Then $Ax \leq b$ is totally dual integral if and only if*

*(i) the rows of $A$ form a Hilbert basis*

*and*

*(ii) for each subset $S$ of inequalities from $Ax \leq b$, if (1) is infeasible, then this can be proven integrally.*

**Solution is available upon request.**

## 4. Combinatorial Optimization

Recall that a graph $G$ is *factor critical* if for all $v \in V(G)$, $G - v$ has a perfect matching. A *near perfect matching* is a matching covering all but one vertex of the graph. It is known that every 2-connected factor critical graph $G$ contains pairwise edge-disjoint subgraphs $G_0, H_1, \ldots, H_k$ satisfying the following. For $j = 1, \ldots, k$, let $G_j = G_0 \cup \bigcup_{i=1}^{j} H_i$.

a. $G_0$ is an odd cycle and $G = G_k$,

b. $H_i$ is an odd length path with both ends in $G_{i-1}$ and no internal vertex in $V(G_{i-1})$. Specifically, the endpoints of $H_i$ are distinct.

You may use this assertion without proof. Show that every 2-connected factor critical graph $G$ contains at least $|E(G)|$ distinct near perfect matchings.

**Solution.** Let $G_0, H_1, \ldots, H_k$ be the decomposition we get from the statement of the problem, and let $G_i = G_0 \cup \bigcup_1^i H_i$. We will inductively show that $G_i$ has $|E(G_i)|$ distinct near perfect matchings for all $i$. As $G_0$ is an odd cycle, the statement clearly holds for $G_0$.

Fix $l \geq 1$, let $m = |E(G_l)|$, and assume $G_l$ has $m$ distinct near perfect matchings. Label the matchings $M_1, \ldots, M_m$. Let $H_{l+1}$ be an odd length path with vertices $v_1, \ldots, v_a$ for some even integer $a$. For every $1 \leq i \leq m$, the near perfect matching $M_i$ can be extended to a near perfect matching $M_i'$ of $G_{l+1}$ by including alternating edges of the path $H_{l+1}$. Note that since $M_i$ must cover at least one of the endpoints $v_1$ and $v_a$ of $H_{l+1}$, it is not the case that both the edges $v_1v_2$

and $v_{a-1}v_a$ are contained in $M_i'$. For every $2 \le i \le a-1$, we can find a perfect matching (call it $N_i$) of $H_{l+1} - v_i$ by using alternating edges from the path $H_{l+1}$ and adding a near perfect matching of $G_l$ avoiding one of the two endpoints of $H_{l+1}$. Note that each $M_i'$ covers every vertex of $H_{l+1}$ except possibly one of the endpoints $v_1$ or $v_a$. Thus, $M_1', \dots, M_m', N_2, \dots, N_{a-1}$ are $m + (a-2) = |E(G_{l+1})| - 1$ distinct near perfect matchings of $G_{l+1}$.

To complete the proof, we need to find one more near perfect matching in $G_{l+1}$. Let $J_1$ be a near perfect matching in $G_l$ covering every vertex but $v_1$. Let $J_2$ be a near perfect matching in $G_l$ covering every vertex except for $v_a$. Then $J_1 \cup J_2$ has components which are even length cycles, single matching edges, and an even length path $P$ from $v_1$ to $v_a$. Let $v'$ be the neighbor of $v_1$ on $P$. By taking alternating edges of $P$ which are not incident to any of the three vertices $v_1$, $v_a$, or $v'$, we see that there exists a matching $J$ in $G_l$ which covers every vertex except $v_1$, $v_a$, and $v'$. By adding alternating edges of $H_{l+1}$, we can extend $J$ to a near perfect matching $J'$ of $G_{l+1}$ covering every vertex except $v'$. Note that by construction, the edges $v_1v_2$ and $v_av_{a-1}$ are both contained in $J'$. Thus, $J'$ is distinct from $M_1', \dots, M_m', N_2, \dots, N_{a-2}$, completing the proof.

## 5. Graph Theory

Prove that for every integer $k \ge 1$ there exists an integer $N$ such that if the subsets of $\{1, 2, \dots, N\}$ are colored using $k$ colors, then there exist disjoint non-empty sets $X, Y \subseteq \{1, 2, \dots, N\}$ such that $X$, $Y$ and $X \cup Y$ receive the same color.
*Hint.* You may want to consider intervals.

**Solution:** By Ramsey's theorem there exists an integer $N$ such that for every $k$-coloring of 2-element subsets of $\{1, 2, \dots, N+1\}$ there exists a 3-element set $A \subseteq \{1, 2, \dots, N+1\}$ such that all 2-element subsets of $A$ receive the same color. We claim that $N$ satisfies the requirements of the problem. For $i, j \in \{1, 2, \dots, N+1\}$ with $i < j$ we color the set $\{i, j\}$ using the color of the set $\{i, i+1, \dots, j-1\}$. By the choice of $N$ there exist $i, j, k \in \{1, 2, \dots, N+1\}$ such that $i < j < k$ and the sets $\{i, j\}$, $\{j, k\}$ and $\{i, k\}$ receive the same color. Then the sets $X := \{i, i+1, \dots, j-1\}$ and $X := \{j, j+1, \dots, k-1\}$ are as desired.

## 6. Probabilistic methods

A proper list-coloring of a graph $G = (V, E)$ from lists $\{L_v \subset \mathbb{N} \mid v \in V\}$ is a function $c : V \to \mathbb{N}$ such that $c(v) \in L_v$ for all $v \in V$ and $c(u) \neq c(v)$ for all $\{u, v\} \in E$.

Let $r$ be a natural number. Prove that if for all $v \in V$ we have $|L_v| = 10r$ and for all $j \in L_v$ there are at most $r$ neighbors $u \in V$ of $v$ such that $j \in L_u$, then $G$ admits a proper list-coloring from these lists.

**Solution:** Consider a random list-coloring $c$ of $G$, where each $c(v)$ is selected from $L_v$ independently and equiprobably. For an edge $e = \{u, v\} \in E$ and a color $j \in L_u \cap L_v$, let $E_e^j$ be the event that $c(u) = c(v) = j$. The event $E_e^j$ is independent of $E_f^i$ when $e$ and $f$ are disjoint or when $j \notin L_{e \cap f}$, so $E_e^j$ is only dependent of at most $d = 2 \cdot (r-1) \cdot 10r$ other events. Since

$$e\,(d+1)\Pr\left[E_e^j\right] = \frac{e\,(20r\,(r-1)+1)}{100r^2} < \frac{e}{5} < 1,$$

by the local lemma, $\Pr\left[\bigcap_{e,j} \overline{E_e^j}\right] > 0$, implying that there is a proper list-coloring of $G$ from the given lists.

## 7. Algebra

Two polynomials $f, g \in R[t]$ over a commutative ring $R$ with identity are called *relatively prime over* $R$ if $f$ and $g$ generate the unit ideal in $R[t]$. Let $f, g \in \mathbf{Z}[t]$ be non-constant monic polynomials such that $f$ and $g$ are relatively prime over $\mathbf{Q}$ and the residues of $f$ and $g$ modulo $p$ are relatively prime over $\mathbf{Z}/p\mathbf{Z}$ for all prime numbers $p$. Prove that $f$ and $g$ are relatively prime over $\mathbf{Z}$.

**Solution:** Since $f$ and $g$ are relatively prime over $\mathbf{Q}$, there exist rational numbers $\alpha, \beta$ such that $\alpha f + \beta g = 1$. Clearing denominators, we find that there exist integers $a, b$ and a positive integer $d$ such that

$$af + bg = d. \tag{4}$$

Without loss of generality, we may assume that $d$ is the *minimal* positive integer for which there is a relation of the form given in (4). We would like to show that $d = 1$.

Suppose for the sake of contradiction that $d > 1$, and let $p$ be a prime number dividing $d$. Then $\bar{a}\bar{f} + \bar{b}\bar{g} = 0$ in $(\mathbf{Z}/p\mathbf{Z})[t]$, which implies that

$$\bar{a}\bar{f} = -\bar{b}\bar{g}. \tag{5}$$

Since $\mathbf{Z}/p\mathbf{Z}$ is a field, $(\mathbf{Z}/p\mathbf{Z})[t]$ is a Unique Factorization Domain, and since $f, g$ are monic and non-constant, $\bar{f}$ and $\bar{g}$ are not units in $(\mathbf{Z}/p\mathbf{Z})[t]$. Thus (5) implies that $\bar{a} = \bar{b} = 0$, which means that $p \mid a$ and $p \mid b$. But then $p \mid d$ as well, and dividing both sides of (4) by $p$ contradicts the minimality of $d$. Thus $d = 1$ as claimed, which means that $f, g$ are coprime over $\mathbf{Z}$.