1. Computability, Complexity and Algorithms

Part a: You are given a graph $G = (V, E)$ with edge weights $w(e) > 0$ for $e \in E$. You are also given a minimum cost spanning tree (MST) T. For one particular edge $e^* = (y, z)$ which is in T, its edge weight is increased (all other edges stay the same).
Specifically the weight of e^* changed from $w(e^*)$ to $\widehat{w}(e^*)$.
Give an algorithm to find a MST for this new edge weighting.
(As fast as possible in $O()$ notation.)

The graph G is given in adjacency list representation and the set T is given as a list of edges, for example as $\{4, 2\}, \{3, 4\}, ...$ You are given the specified edge $e^* = (y, z)$ and you are given the old weights $w()$ for all edges of G, and the new weight $\widehat{w}(e^*)$.

Part b: Suppose you have a flow network $G = (V, E)$ with integer capacities $c_e > 0$ for $e \in E$, and you are given a maximum flow f^* from s to t. Let C^* denote the size of this flow f^*. Now suppose that for one particular edge e^* we decrease the capacity of e^* by one, from c_{e^*} to $c_{e^*} - 1$. Give an algorithm to output a maximum flow in the new graph. (As fast as possible in $O()$ notation.)

Solution:
Part a: The algorithm:

1. Remove e^* from T.
2. The graph $T \setminus \{e^*\}$ has two components. Run DFS in $T \setminus \{e^*\}$ to label the vertices with their respective components.
3. Go through all of the edges of G to find the minimum weight edge e' that crosses between the 2 components of $T \setminus \{e^*\}$.
4. Return $T' = T \cup \{e'\} \setminus \{e^*\}$.

The algorithm takes $O(|E|)$ time.
Correctness: Each of the edges of $T \setminus \{e^*\}$ were minimum across some cut in the original graph to be part of the MST T. In the new graph they are still minimum across the same cut. Hence, $T \setminus \{e^*\}$ is part of a MST in this new graph. Thus, since e' is also minimum weight across a cut, by the cut property again, T' is a MST.
Running time: Step 2 takes $O(n)$ time and then step 3 takes $O(1)$ time per edge so a total time of $O(m)$.
Part b: If $f_{e^*} < c_{e^*}$ then f^* is still a valid flow in the new graph and so we simply output f^*. If the edge e^* was fully capacitated then we do the following. We first need to find a path $\mathcal{P}$ with positive flow that goes through e^*. Let $e^* = (v, w)$. Let G' be the graph of edges with positive flow. Then run DFS on G' to find a path $\mathcal{P}_1$ from v to s and then run DFS to find a path $\mathcal{P}_2$ from w to t. If these two paths do not share an edge then let $\mathcal{P} = \mathcal{P}_1 \cup e^* \cup \mathcal{P}_2$. Decrease the

flow in f^* by 1 unit along $\mathcal{P}$. For the resulting flow, build the residual network and look for an augmenting path. Output the resulting flow.
If the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ share an edge then that means there is a cycle $\mathcal{C}$ containing e^* with positive flow. We simply decrease the flow in f^* by 1 unit along C and this gives a max flow in the new graph.

2. Analysis of Algorithms

Let $G = (V, E)$ be a directed graph with source s, sink t and capacities on edges. Give a polynomial time algorithm for deciding if G has a unique minimum s–t cut.

Solution: Find a max s-t flow, f, in G and construct the residual graph $R(G, f)$. R will have paths from t to s but no paths from s to t. Next, find the Picard-Queyranne structure by finding strongly connected components in R and shrinking them. This will yield a DAG. Let Q_s and Q_t be the components containing s and t, respectively. The ideal cuts between these two vertices are in one-to-one correspondence with the set of minimum s-t cuts in G.

Now G has a unique minimum s-t cut iff there is only one ideal cut. This is easy to check in polynomial time.

3. Theory of Linear Inequalities

Let $P \subseteq [0, 1]^n$ be an integral polytope contained in the 0/1 cube with ℓ_1-diameter bounded by k, i.e., the polytope's vertices have only entries in $\{0, 1\}$ and $\max_{x,y \in P} \|x - y\|_1 \leq k$. The goal is to maximize an objective $c \in \mathbb{Z}^n$ over P. Without loss of generality you may assume that $c \geq 0$ as we can simply flip the coordinates of the cube. You are given a feasible integral solution $x_0 \in P$ and access to the polytope P is restricted to querying the following oracle:

Augmentation oracle $O(x_0, c)$:
Input: $x_0 \in P$ integral, objective $c \in \mathbb{Z}^n$
Output: $x \in P$ integral with

$$cx > cx_0,$$

if such an x exists, otherwise return OPTIMAL.

Let $C := \|c\|_\infty$ and let $K := \lfloor \log C \rfloor$. Define the following sequence of objective functions $c^k := \lfloor c/2^{K-k} \rfloor$ (coordinate-wise operation) and consider the following bit scaling algorithm:

1. Repeat for $k = 0, \dots, K$
   (a) While x_k is not *OPTIMAL* for c^k do
      i. $x_k \leftarrow O(x_k, c^k)$
   (b) $x_{k+1} \leftarrow x_k$.
2. Return x_{K+1}.

Task.

Prove that the algorithm optimizes c over P with at most $(K+1)k$ oracle calls.

Solution: Clearly, x_{K+1} is the optimal solution as $c^K = c$. Moreover, the number of outer iterations is bounded by $K+1$. Thus it remains to verify that the number of inner iterations is bounded by k.

To this end observe that at the end of (outer) iteration $k-1$, we have that x_k is an optimal solution w.r.t. c^{k-1}. Now we can write $c^k = 2c^{k-1} + c(k)$ for some 0/1 vector $c(k)$. If now x_{k+1} is the optimal solution for c^k, we have

$$c^k(x_{k+1} - x_k) = 2 \underbrace{c^{k-1}(x_{k+1} - x_k)}_{\leq 0, \text{ as } x_k \text{ opt. for } c^{k-1}} + \underbrace{c(k)(x_k - x_{k+1})}_{\leq \|x_k - x_{k+1}\| \leq k} \leq k,$$

and hence the claim follows.

4. Combinatorial Optimization

We are given an undirected graph $G = (V, E)$ and every edge has a color. This is represented by a partition of E into $E_1 \cup \ldots \cup E_k$ where each E_i represents a set of edges of the same color i. A spanning tree is called bi-colorful if it contains at most two edges of any color.

1. Give an efficient algorithm that checks whether there is a a bi-colorful spanning tree in G and show its correctness.

2. Show that a graph G has a bi-colorful spanning tree if and only if for any disjoint set of colors $I, J \subseteq \{1, \ldots, k\}$ and any $F \subseteq \cup_{i \in I \cup J} E_i$ such that $|F \cap E_i| = 1$ for each $i \in I$ and $F \supseteq E_i$ for each $i \in J$, $G \setminus F$ has at most $|I| + 2|J| + 1$ components.

Solution:

1. Consider the following two matroids. The first matroid $\mathcal{M}_1 = (E, \mathcal{I}_1)$ is the graphic matroid. The second matroid $\mathcal{M}_2 = (E, \mathcal{I}_2)$ is the partition matroid where $F \in \mathcal{I}_2$ if and only if $|F \cap E_i| \leq 2$ for each $1 \leq i \leq k$. Observe that a graph has a bi-colorful spanning tree if and only if there is a common independent set of size $|V| - 1$. Thus one can use matroid intersection to check whether a graph has a colorful spanning tree.

2. We first prove the necessity. Consider any disjoint set of colors $I, J \subseteq \{1, \ldots, k\}$ and any $F \subseteq E$ such that $|F \cap E_i| = 1$ for each $i \in I$, $F \supseteq E_i$ for each $i \in J$ and $F \cap E_i = \emptyset$ for each $i \notin I \cup J$,. Then any bi-colorful tree can pick at most one arc from $F \cap E_i$ for each $i \in I$ and at most two arcs from $E_i \supseteq F \cap E_i$ for each $i \in J$. Thus bi-colorful tree can pick at most $|I| + 2|J|$ edges from F. Thus $G \setminus F$ can have at most $|I| + 2|J| + 1$ components.

Now, we argue sufficiency. Let r_1 and r_2 denote the rank functions of the $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Observe that for any $F \subseteq E$, $r_1(F) = |V| - \kappa(F)$ where $\kappa(F)$ denotes the number of connected components in (V, F). Also, $r_2(F) = \sum_{i=1}^{k} \min\{2, |F \cap E_i|\}$. From part (a), G has a bi-colorful spanning tree if the maximum common independent set has size at least $|V| - 1$. But the size of the maximum independent set equals $\min_{F \subseteq E} r_1(E \setminus F) + r_2(F)$. We now show this minimum is at least $|V| - 1$. Let F to be the minimizer. If $|F \cap E_i| \geq 2$, then we can assume that $F \supseteq E_i$. This follows since updating F to $F \cup E_i$ does not change $r_2(F)$ but can only decrease $r_1(E \setminus F)$. Let $I = \{i : |F \cap E_i| = 1\}$ and $J = \{i : |F \cap E_i| \geq 2\}$. Then $r_2(F) = |I| + 2|J|$ and $\kappa(E \setminus F) \leq |I| + 2|J| + 1$ by the assumption. Thus $r_1(E \setminus F) \geq |V| - 1 - |I| - 2|J|$ and $r_1(E \setminus F) + r_2(F) \geq |V| - 1$ as required.

5. Graph Theory

Let $n \geq 1$ be an integer. At a round table there are $2n$ Canadians, $2n$ Americans and $2n$ Mexicans. The people whose neighbors are of the same nationality are asked to stand up. What is the largest possible number of people that can be asked to stand up?
Note: For example, for an American to stand up his two neighbors must be of the same nationality, but not necessarily American.

Solution: The number is $6n - 4$. To see that consider a graph with one vertex per person and an edge if the two persons are seated with exactly one person between them. What we get is the union of two disjoint cycles of length $3n$ (the persons seated in an even position and the ones seated in an odd position). We color the vertices with 3 colors according to nationality. Note that, if there are two adjacent vertices of the same color, then the person between them is standing up, and if not the person between them remains seated. Thus we want to maximize the number of adjacent vertices of the same color. Consider one of the cycles. As there are only $2n$ vertices of each color, it cannot be monochromatic. Consider a vertex v_0 in that cycle. Start in v_0 and go around the cycle until you reach a vertex of another color. You can do this in two ways, one for each direction in the cycle. Note that when you find a vertex of different color for the first time you use a different edge depending on which sense you chose to go along the cycle.

Thus in that cycle we have at least 2 seated persons. If we do the same for the other cycle we get that at most $6n - 4$ persons are standing up. To see that this number can be reached, seat the $2n$ Canadians in $2n$ consecutive vertices of one of the cycles, the $2n$ Americans in $2n$ consecutive vertices of the other cycle and the $2n$ Mexicans in the remaining seats. Then we obtain an arrangement where only 4 persons stay seated.

6. Probabilistic methods

Let $T = T(n, p)$ be the random (complete) binary tree of depth n (that is, it has 2^n leaves in total), where each edge is present with probability p. Let X_i be a random indicator variable for the reachability of the ith leaf from the root and denote by

$$X = \sum_{i=1}^{2^n} X_i$$

the number of reachable leaves from the root. Use the second moment method in order to show that for the threshold $p > 1/2$, one has

$$\mathbf{Prob}[X > 0] \ > \ 0.$$

Moreover, the lower bound on this probability is a constant that depends on p.
Hint: You can use the fact that $\mathbf{Prob}[X > 0] \geq \frac{(\mathbf{E}[X])^2}{\mathbf{E}[X^2]}$. (You do not need to prove this property.)

Solution: We have $\mathbf{E}[X_i] = p^n$, for $i = 1, \ldots, 2^n$. Therefore

$$\mathbf{E}[X] = 2^n p^n,$$

which goes to ∞ as $n \to \infty$, under the assumption $p > 1/2$.

We are now using the fact that

$$\mathbf{Prob}[X > 0] \geq \frac{(\mathbf{E}[X])^2}{\mathbf{E}[X^2]},$$

and thus we are left to bound $\mathbf{E}[X^2]$. We have

$$X^2 = \left(\sum_{i=1}^{2^n} X_i\right)^2 = \sum_{i=1}^{2^n} X_i^2 + \sum_{1 \leq i \neq j \leq 2^n} X_i X_j.$$

Let $k(i,j)$ be the depth of the node at which the two respective paths from the root to leaves i and j split. Then

$$\mathbf{E}[X_i X_j] = p^{2n-k(i,k)}.$$

Thus:

$$\sum_{1 \leq i \neq j \leq 2^n} X_i X_j = \sum_{1 \leq i \neq j \leq 2^n} p^{2n-k(i,k)} = \sum_{k=0}^{n-1} 2^{2n-k-1} p^{2n-k}.$$

The latter is equal to:

$$\frac{1}{2} \sum_{k=0}^{n-1} 2^{2n} \frac{1}{2^k} p^{2n} \frac{1}{p^k} = \frac{1}{2}(2p)^{2n} \sum_{k=0}^{n-1} \frac{1}{(2p)^k} \leq \frac{1}{2}(2p)^{2n} \frac{2p}{2p-1} = \frac{p}{2p-1}(2p)^{2n}.$$

Therefore

$$\mathbf{E}[X^2] \leq \frac{p}{2p-1}(2p)^{2n} + 2^n p^n = \frac{p}{2p-1}(2p)^{2n}(1 + o(1)).$$

Overall, we obtain:

$$\mathbf{Prob}[X > 0] \geq \frac{(\mathbf{E}[X])^2}{\mathbf{E}[X^2]} \geq \frac{2p-1}{p}(1 - o(1)),$$

and the latter is indeed a constant that depends on p, assuming $p > 1/2$.

7. Algebra

Let ω be the complex number $\omega = e^{2\pi i/3}$ and let i be the complex number $i = e^{2\pi i/4}$. Which of the following rings are isomorphic?

1. $\mathbb{Z}[\omega]/ < 23 >$
2. $\mathbb{Z}[i]/ < 23 >$
3. $\mathbb{Z}/529$
4. $\mathbb{Z}/23 \times \mathbb{Z}/23$

Solution: We first claim that 1) and 2) are both isomorphic to the finite field with $23^2 = 529$ elements: Since ω is a primitive 3rd root of unity, its minimal polynomial is $x^2 + x + 1$. Thus $\mathbb{Z}[\omega] = \mathbb{Z}[x]/ < x^2 + x + 1 >$. Thus $\mathbb{Z}[\omega]/ < 23 >= \mathbb{Z}/23[x]/ < x^2 + x + 1 >$. Since $(\mathbb{Z}/23)^* \cong \mathbb{Z}/22$, there is no primitive 3rd root of unity in $\mathbb{Z}/23$. Moreover, the polynomial $x^2 + x + 1$ is separable over $\mathbb{Z}/23$ because it is a factor of the separable polynomial $x^3 - 1$. (You can see $x^3 - 1$ is separable by computing that its derivative is $3x^2$, which is non-zero and shares no common factor with $x^3 - 1$.) It follows that $x^2 + x + 1$ has no roots in $\mathbb{Z}/23$ and is thus irreducible, prime, and maximal in $\mathbb{Z}/23[x]$. Thus $\mathbb{Z}[\omega]/ < 23 >$ is indeed a field. Moreover, $\mathbb{Z}/23[x]/ < x^2 + x + 1 >$ has 23^2 elements corresponding to linear polynomials $ax + b$ with a, b running from 0 to 23.

The analysis of $\mathbb{Z}[i]/ < 23 >$ is similar. The minimal polynomial of i is $x^2 + 1$, which is also irreducible in $\mathbb{Z}/23[x]$ because $(\mathbb{Z}/23)^* \cong \mathbb{Z}/22$ has no primitive fourth root of unity. The rest of the argument is the same.

Since 1) and 2) are fields and 3) and 4) have zero divisors, 1) and 2) are not isomorphic to 3) or 4).

3) and 4) themselves represent district isomorphism classes because 3) has an element of order 529 and for any element (a, b) of $\mathbb{Z}/23 \times \mathbb{Z}/23$ we have that $23(a, b) = 0$.