1. Computability, Complexity and Algorithms

Given an undirected graph $G = (V, E)$ with $n$ vertices, two vertices $s, t \in V$ and an integer $N$, the #paths problem is to determine whether there exist at least $N$ distinct $s$-$t$ simple paths in $G$ (note we say distinct, not disjoint). Use the following steps to show the problem is NP-complete by a reduction from the Hamiltonian cycle problem.

1. Show that the number of simple cycles through a given edge of a given graph $G$ can be counted by a reduction to the #paths problem.

2. Given a simple undirected graph $H = (U, F)$, let $H'$ be obtained by subdividing each edge into $\ell$ edges and creating $k$ parallel copies of each edge. Take a single Hamiltonian cycle in $H$. How many distinct Hamiltonian cycles in $H'$ does it map to?

3. Suppose $H$ is Hamiltonian and $H'$ is constructed with $k = n$, $\ell = n+c$. Show that the total number of non-Hamiltonian simple cycles in $H'$ is smaller than the number of Hamiltonian cycles in $H'$ by a factor of $n^c$.

4. Show that the problem of deciding whether a given graph has a Hamiltonian cycle can be reduced to the #paths problem in polynomial time.

2. Theory of Linear Inequalities

Let $x^*$ be a fractional extreme point of a rational polytope $P := \{ x \in \mathbb{R}^n \mid Ax \leq b \}$. Prove that there exists a Chvátal-Gomory cut for $P$ that separates $x^*$.

3. Graph Theory

Let $G$ be a 2-connected plane graph and let $V(G), E(G), F(G)$ denote its set of vertices, set of edges, set of faces, receptively. Let $\sigma : V(G) \cup F(G) \rightarrow \mathbb{Z}$ such that $\sigma(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$, where $d(x)$ is the number of edges incident with $x$. Show that

1. $\sum_{x \in V(G) \cup F(G)} \sigma(x) = -8$.

2. If $\delta(G) \geq 5$ then $G$ contains $K_4^-$ (obtained from $K_4$ by removing an edge) as a subgraph.

4. Algebra

(a) Let $R$ be an integral domain containing a field $k$ as a subring. Suppose that $R$ is a finite dimensional vector space over $k$ under the ring multiplication. Show that $R$ is a field.
(b) Show that the conclusion does not hold without the assumption of being finite dimensional. That is, give an example of a ring $R$ containing a field $k$ as a subring, such that $R$ is an integral domain but is not a field.

(c) Show that the conclusion does not hold without the assumption of being an integral domain. That is, give an example of a ring $R$ containing a field $k$ as a subring, such that $R$ is finite dimensional over $k$ but is not a field.

4. Linear Algebra

Let $A, B$ be $n \times n$ matrices. Show that $\sigma(AB) = \sigma(BA)$. (Recall the spectrum of $A$, $\sigma(A) = \{\lambda : A - \lambda I \text{ is not invertible}\}$.)

5. Analysis of Algorithms

Part a: Let $f(x)$ be a real-valued function. You are given a randomized algorithm $B$ that, given input $x$ and parameter $\epsilon > 0$, outputs $B(x)$ which approximates $f(x)$ as follows:

$$\forall x, \quad \Pr[(1-\epsilon)f(x) \leq B(x) \leq (1+\epsilon)f(x)] \geq 3/4. \quad (1)$$

Give an algorithm $C$ that, given input $x$ and parameters $\epsilon, \delta > 0$, outputs $C(x)$ satisfying:

$$\forall x, \quad \Pr[(1-\epsilon)f(x) \leq C(x) \leq (1+\epsilon)f(x)] \geq 1 - \delta. \quad (2)$$

Achieve the best dependence on $\delta$ in $O(\cdot)$ notation (i.e., ignore constant factors).

Part b: Explain if your approach in (a) still works if instead of (??) the probability of success is weakened so that:

$$\forall x, \quad \Pr[(1-\epsilon)f(x) \leq B(x) \leq (1+\epsilon)f(x)] \geq 1/4. \quad (3)$$

Part c: Suppose that we have $n$ polygons $P_1, \ldots, P_n$, all lying inside $[0, 1] \times [0, 1]$ which is the square with side length 1 on the Euclidean plane. Every polygon has area at least $\alpha > 0$. You are not given the polygons explicitly but instead for each polygon $P_i$ we have access to a membership oracle: given a point $x \in \mathbb{R}^2$, the oracle returns YES if $x \in P_i$ and NO if $x \notin P_i$.

Give a randomized algorithm that approximately estimates the area of the union of these polygons. Given $0 < \varepsilon < 1$, the output $Y$ of your algorithm should satisfy

$$\Pr[(1-\varepsilon)|P| \leq Y \leq (1+\varepsilon)|P|] \geq \frac{3}{4}$$

where $|P|$ denotes the area of $P = P_1 \cup \cdots \cup P_n$.

(You may assume that sampling a real number uniformly at random from $[0, 1]$ takes constant time, and that each oracle call takes constant time.)

The running time of your algorithm should be polynomial in $n, 1/\varepsilon$, and $1/\alpha$. 
6. Combinatorial Optimization

Recall that a graph $G$ is factor-critical if for all $v \in V(G)$, $G - v$ has a perfect matching. An open odd ear decomposition of $G$ is a sequence $H_0, H_1, \ldots, H_k$ of subgraphs of $G$ such that, letting $G_j = \bigcup_{i=0}^{j} H_i$ for $j = 0, 1, \ldots, k$, we have

(a) $G_0$ is an odd cycle,

(b) for $i = 1, 2, \ldots, k$ the graph $H_i$ is an odd length (i.e., odd number of edges) path with both (distinct) end vertices in $V(G_{i-1})$ and no internal vertex or edge in $V(G_{i-1})$, and

(c) $G = G_k$.

Show that if a 2-connected graph $G$ is factor-critical, then it admits an open odd ear decomposition.

7. Probabilistic methods

Prove that there is some constant $c > 0$ so that for every integer $k \geq 1$, given a graph and a set of $k$ acceptable colors for each vertex such that every color is acceptable for at most $ck$ neighbors of each vertex, there is always a proper coloring where every vertex is assigned one of its acceptable colors. (Recall as usual that a proper coloring requires that the endpoints of every edge get different colors.)