6-CONNECTED GRAPHS ARE TWO-THREE LINKED

A Dissertation
Presented to
The Academic Faculty

By

Shijie Xie

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in
Algorithms, Combinatorics and Optimization

School of Mathematics
Georgia Institute of Technology

October 2019

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Approved by:

Dr. Xingxing Yu, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Robin Thomas
School of Mathematics
Georgia Institute of Technology

Dr. Prasad Tetali
School of Mathematics
Georgia Institute of Technology

Dr. Richard Peng
School of Computer Science
Georgia Institute of Technology

Dr. Lutz Warnke
School of Mathematics
Georgia Institute of Technology

Date Approved: October 24, 2019
No, emptiness is not nothingness. Emptiness is a type of existence. You must use this existential emptiness to fill yourself.

*Liu Cixin, The Three-Body Problem*
To my parents and my wife.
ACKNOWLEDGEMENTS

It has been wonderful to be at Georgia Tech for four years, and there are lots of great memories here. Now, it is time for me to say goodbye and start a new journey, but I really owe a lot of thanks to people I know.

There are many people and things coming to my mind when I am writing the acknowledgements. Undoubtedly, the first person on the list is my advisor, Prof. Xingxing Yu. This dissertation would not have been possible without his careful support. I would like to express my deepest gratitude to him, for his knowledgeable mentorship, for his interesting thoughts, for his persistent guidance, for his useful advice, for his sincere care, and for his inspirational encouragement. I am extremely fortunate to have him as my advisor. I really enjoyed and will truly miss our discussion time, when I was led to an exciting structured graph theory world and attempted to solve different complicated conjectures under his instruction. More importantly, his spirit of perseverance and his courage in the face of difficulties will always inspire me.

I am grateful to Prof. Robin Thomas, as well as the ACO program for providing me an important Ph.D. offer and a valuable chance to do research in combinatorics with so many amazing faculty members and peers. Moreover, I must thank Prof. Robin Thomas very much for his careful guidance and for providing me conference and research assistant opportunities. It is also awesome to take his graph theory class and be his coworker.

I would also like to thank Prof. Richard Peng, Prof. Prasad Tetali, Prof. Robin Thomas, Prof. Lutz Warnke, and Prof. Xingxing Yu for serving on my dissertation committee. I thank Prof. Gexin Yu for carefully reading on my thesis and for taking on the role of reader. I want to thank all the professors who have taught me classes at Georgia Tech. I also want to thank Klara Grodzinsky, Morag Burke, and all other School of Mathematics staffs for their kind assistance and support.

I am so fortunate and happy to discuss problems with Xiaofan Yuan, Qiqin Xie, Yan
Wang, Le Liang, Changong Li, and Dawei He. I learned a lot from them, and I am very grateful to them for their helps. I may no longer have a chance to meet them anymore, but I truly wish them the best of luck in the future.

I am also very happy to make many other friends at School of Mathematics who made my Ph.D. life more colorful, including Tongzhou Chen, He Guo, Ruilin Li, Shu Liu, Jiaqi Yang, Dantong Zhu, Kisun Lee, Yuze Zhang, Yian Yao, Xin Xing, Youngho Yoo, and many others. I am not able to list all of them, but I treasure all the fun moments with them.

Last but not least, my deepest gratitude goes to my wife and my parents for their endless love, support and encouragement. Without them, I would not be able to move forward. To them, I dedicate this thesis.
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SUMMARY

Let $G$ be a graph and $a_0, a_1, a_2, b_1,$ and $b_2$ be distinct vertices of $G$. Motivated by their work on Jørgensen’s conjecture, Robertson and Seymour asked when does $G$ contain disjoint connected subgraphs $G_1, G_2$, such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. We prove that if $G$ is 6-connected then such $G_1, G_2$ exist. Joint work with Robin Thomas and Xingxing Yu.
CHAPTER 1  
INTRODUCTION AND BACKGROUND

The Four Color Theorem [1, 2, 3] asserts that every loopless planar graph admits a vertex 4-colouring. The related problem was first put forward by Francis Guthrie in 1852, who asked whether it was true that any planar map can be colored with four colors such that adjacent regions receive different colors. In 1976, Appel and Haken [1] claimed a proof of the Four Color Theorem with the help of a computer. However, some computer-free parts of their proof are complicated and tedious to verify. In 1997, Robertson, Sanders, Seymour, and Thomas [2, 3] gave a much simpler proof for the Four Color Theorem.

According to Kuratowski’s theorem [4], a graph is planar if and only if it contains no $K_5$-subdivision or $K_{3,3}$-subdivision. Moreover, it is well known that any 3-connected nonplanar graph other than $K_5$ contains a $K_{3,3}$-subdivision. Hence, as an extension of the Four Color Theorem, it is natural to ask whether every graph without $K_5$-subdivision is also 4-colorable. More generally, Hajós [5] conjectured that for any positive integer $k$, every graph containing no $K_{k+1}$-subdivision is $k$-colorable. This conjecture is true for $k \leq 3$, but Catlin [5] found counterexamples to this conjecture for each $k \geq 6$. However, the cases for $k = 4$ and $k = 5$ are still open. Efforts have been made to resolve Hajós’ conjecture for $k = 4$. Yu and Zickfeld [6] proved that a minimum counterexample to Hajós’ conjecture when $k = 4$ must be 4-connected. Moreover, Sun and Yu [7] showed that if $G$ is a minimum counterexample to Hajós’ conjecture and $S$ is a 4-cut in $G$ then $G - S$ has exactly two components. In fact, if one can show a minimum counterexample to Hajós’ conjecture for $k = 4$ is 5-connected, then Hajós’ conjecture for $k = 4$ will immediately follow from the Kelmans-Seymour conjecture [8, 9]: Every 5-connected nonplanar graph contains $K_5$-subdivision. This Kelmans-Seymour conjecture was recently proved by He, Wang, and Yu [10, 11, 12, 13].
While Hajós’ conjecture concerns the chromatic number of graphs without $K_{k+1}$-subdivision, Hadwiger [14], in 1943, conjectured a far-reaching generalization of the Four Color Theorem in terms of $K_{k+1}$-minor: For any positive integer $k$, if a graph contains no $K_{k+1}$-minor then it is $k$-colorable.

It is easy to prove that Hadwiger’s conjecture holds for $k \leq 2$. Hadwiger [14] and Dirac [15] proved the case for $k = 3$. For $k = 4$, Hadwiger’s conjecture is equivalent to the Four Color Theorem by the result of Wagner [16], which characterized graphs containing no $K_5$-minor and showed that Four Color Theorem implies that graphs containing no $K_5$-minor are 4-colorable. The case $k = 5$ can also be reduced to the Four Color Theorem, as shown by Robertson, Seymour, and Thomas [17]. However, this conjecture remains open for $k \geq 6$.

In fact, there are also many other interesting results related to Hadwiger’s conjecture. Suppose Hadwiger’s conjecture is false for some $k$, and let $G$ be a minor minimal counterexample. Dirac [15] showed that $G$ is 5-connected when $k \geq 5$, and Mader [18] showed that $G$ is 6-connected when $k \geq 5$, and 7-connected when $k \geq 6$. Kawarabayashi and G. Yu [19] proved that $G$ is $(2k/27)$-connected, improving upon an earlier bound in [20].

Let the *stability number* $\alpha(G)$ of a graph $G$ denote the size of the largest stable set. Then every $n$-vertex graph $G$ has chromatic number at least $\lceil n/\alpha(G) \rceil$, and should contain a clique minor of this size if Hadwiger’s conjecture is true. In 1982, Duchet and Meyniel [21] proved that every $n$-vertex graph $G$ has a $K_k$ minor where $k \geq n/(2\alpha(G) - 1)$. Moreover, there has been a subsequent improvement by Fox [22]. And then Balogh and Kostochka [23] further improved the result, and showed that every $n$-vertex graph $G$ has a $K_k$ minor where $k \geq 0.51338n/\alpha(G)$. Later, in 2007, Kawarabayashi and Song [24] proved that every $n$-vertex graph $G$ with $\alpha(G) \geq 3$ has a $K_k$ minor where $k \geq n/(2\alpha(G) - 2)$.

For an $n$-vertex graph $G$ with $\alpha(G) = 2$, the Duchet-Meyniel theorem implies that there is a $K_k$ minor with $k \geq n/3$, which was strengthened by Böhme, Kostochka and Thomason [25] in 2011. They proved that every $n$-vertex graph with chromatic number $t$
has a $K_k$ minor where $k \geq (4t - n)/3$.

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. So graphs with stability number two are claw-free. Fradkin [26] showed that every $n$-vertex connected claw-free graph $G$ with $\alpha(G) \geq 3$ has a $K_k$ minor where $k \geq n/\alpha(G)$. Furthermore, in 2010, Chudnovsky and Fradkin [27] proved that every claw-free graph $G$ with no $K_{k+1}$ minor is $\lceil 3k/2 \rceil$-colourable.

Since line graphs are claw-free, these results about claw-free graphs are related to a theorem of Reed and Seymour. They showed [28] that Hadwiger’s conjecture is true for line graphs (of multigraphs).

We say that $H$ is an *odd minor* of $G$ if $H$ can be obtained from a subgraph $G'$ of $G$ by contracting a set of edges that is a cut of $G'$. Clearly, a graph contains $K_3$ as an odd minor if and only if it is not 2-colourable. In 1979, Catlin [5] showed that if $G$ has no $K_4$ odd minor then $G$ is 3-colourable.

A *fully odd* $K_4$ in $G$ is a subgraph of $G$ which is obtained from $K_4$ by replacing each edge of $K_4$ by a path of odd length in such a way that the interiors of these six paths are disjoint. Then in 1998, Zang [29] proved (and, independently, Thomassen [30] proved in 2001) the conjecture of Toft [31] that if $G$ contains no fully odd $K_4$ then $G$ is 3-colourable.

Moreover, in 1995, Gerards and Seymour conjectured a strengthening of Hadwiger’s conjecture (see [32]) that for every $k \geq 0$, if $G$ has no $K_{k+1}$ odd minor, then $G$ is $k$-colourable, and it is known as true for $k \leq 3$.

In fact, one can find more interesting results and open problems about Hadwiger’s conjecture and its variations from a survey [33], written by Seymour in 2016.

Now, we just go back and spend a bit more space on the $k = 5$ case of the Hadwiger conjecture. As we mentioned, Mader [18] proved that any minor minimal counterexample to the Hadwiger conjecture for $k = 5$ is 6-connected. Jørgensen [34] conjectured that every 6-connected graph contains a $K_6$-minor or has a vertex whose removal results in a planar graph. Therefore, if Jørgensen’s conjecture holds, then Hadwiger’s conjecture for $k = 5$
easily reduces to the Four Color Theorem. In 2017, Kawarabayashi, Norine, Thomas, and Wollan [35] showed that Jørgensen’s conjecture holds for sufficiently large graphs.

In their work [17], Robertson, Seymour, and Thomas proved that Jørgensen’s conjecture holds for each 6-connected graph in which some edge is contained in four triangles. (However, they were not able to resolve the Jørgensen conjecture. Instead, they explored different structures of a minimum counterexample to the Hadwiger conjecture.) It is natural and useful to extend this result to graphs in which some edge is contained in three triangles: Given a 6-connected graph $G$ and triangles $a_i b_1 b_2 a_i$ for $i = 0, 1, 2$ in $G$, can we prove that $G$ contains $K_6$ minor or has a vertex whose removal results in a planar graph?

A first step is to prove that 6-connected graphs are two-three linked: If $G$ is a 6-connected graph and $a_0, a_1, a_2, b_1, b_2$ are distinct vertices of $G$, then $G$ contains disjoint connected subgraphs $G_1, G_2$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. We believe that we have such a characterization which is quite complicated (even to state) and its proof is long.

For convenience, we use $(G, a_0, a_1, a_2, b_1, b_2)$ to denote a graph $G$ and distinct vertices $a_0, a_1, a_2, b_1, b_2$ of $G$, and call it a rooted graph. A cluster in a graph $G$ is a set $\mathcal{X}$ of disjoint subsets of $V(G)$ such that each member of $\mathcal{X}$ induces a connected subgraph of $G$. We say that a rooted graph $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible if there exists a cluster $\{X_1, X_2\}$ in $G$ such that $\{a_0, a_1, a_2\} \subseteq X_1$ and $\{b_1, b_2\} \subseteq X_2$. We can now state our result as follows.

**Theorem 1.0.1** Let $(G, a_0, a_1, a_2, b_1, b_2)$ be a rooted graph, and assume $G + b_1 b_2 + \{a_i b_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ is 6-connected. Then $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible.

We may view the problem of characterizing feasible rooted graphs as a generalization of the following problem of characterizing 2-linked graphs: Given a graph $G$ and four distinct vertices $a_1, a_2, b_1, b_2$ of $G$, when does $G$ contain disjoint paths from $a_1, a_2$ to $b_1, b_2$, respectively? Several characterizations of 2-linked graphs are given in [36, 37, 38, 39]
and have been used extensively in the literature for proving important structural results on graphs (e.g., in the graph minors project of Robertson and Seymour).

On a high level of the proof, we will always assume that \( \gamma := (G, a_0, a_1, a_2, b_1, b_2) \) is a given rooted infeasible graph such that \( b_1b_2 \notin E(G) \), \( a_ib_j \notin E(G) \) for \( i = 0, 1, 2 \) and \( j = 1, 2 \), and \( G^* := G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\} \) is 6-connected.

Then in Chapter 2, we show that \( G \) has a frame \( A, B \) with respect to \( a_i \) for some \( i \in \{0, 1, 2\} \) in \( (G, a_0, a_1, a_2, b_1, b_2) \), where \( A, B \) are disjoint paths in \( G - a_i \) from \( a_{i-1}, b_1 \) to \( a_{i+1}, b_2 \), respectively (with \( a_{-1} = a_2, a_3 = a_0 \)). We say that a \( B \)-bridge of \( G \) is a subgraph of \( G \) induced by all edges in a component of \( G - V(B) \) and all edges from that component to \( B \). Given a frame \( A, B \) w.r.t. \( a_i \) for some \( i \in \{0, 1, 2\} \), we can prove that the \( B \)-bridge of \( G \) containing \( a_i \) has a disk representation with \( B, a_i \) occurring on the boundary of the disk. Moreover, we define a doublecross in frame \( A, B \), and prove that \( A, B \) has no doublecross.

These properties make the structure of \( G \) much simpler and clearer, but it is still not enough. Hence, in Chapter 3, we need to produce good frames and ideal frames \( A, B \) w.r.t. \( a_i \) for some \( i \in \{0, 1, 2\} \) in \( G \) (with desired nice properties, such as the \( B \)-bridge of \( G \) containing \( a_i \) is maximal). We also divide the \( (A \cup B) \)-bridges of \( G \) between \( A \) and \( B \) into slim connectors and fat connectors. Then our proof is split into two cases: when there does not exist any fat connector in any ideal frame \( A, B \), which is solved in Chapter 6, and when there exists at least one fat connector in some ideal frame \( A, B \), which is solved in Chapter 4 and 5.

For the case without any fat connector, \( G - V(A) \) has a disk representation with \( B \) and \( a_0 \) on the boundary of the disk, and any \( A-B \) path in \( G \) is induced by a single edge. So the structure of \( G \) is quite simple in some sense. For the second case, the structure is more complicated, where an \( A-B \) path in \( G \) is not just a single edge, and different \( A-B \) paths may intersect with each other. However, in both cases, we will try to find a configuration with special properties, which may help us force a small cut in \( G \) or show that \( (G, a_0, a_1, a_2, b_1, b_2) \) is feasible.
Finally, we end this chapter with some notation and terminology. Let $G_1, G_2$ be two graphs. We use $G_1 \cup G_2$ (respectively, $G_1 \cap G_2$) to denote the graph with vertex set $V(G_1) \cup V(G_2)$ (respectively, $V(G_1) \cap V(G_2)$) and edge set $E(G_1) \cup E(G_2)$ (respectively, $E(G_1) \cap E(G_2)$). Let $G$ be a graph, a separation in $G$ is a pair $(G_1, G_2)$ of edge-disjoint subgraphs $G_1, G_2$ of $G$ such that $G = G_1 \cup G_2$. And $|V(G_1) \cap V(G_2)|$ is the order of the separation $(G_1, G_2)$.

Let $P$ be a path, and let $u, v \in V(P)$. Then $P[u, v] := P[u, v] - v$, $P(u, v) := P[u, v] - u$, and $P(u, v) := P[u, v] - \{u, v\}$. Let $B$ be a subgraph of a graph $G$. Then a $B$-bridge of $G$ is a subgraph of $G$ induced by all edges in a component of $G - V(B)$ and all edges from that component to $B$. 
In this chapter, we state some known results and prove some lemmas that we will use. In particular, we show that an infeasible rooted graph must contain a "frame" which consists of two disjoint paths.

A result we use often is Seymour’s characterization of 2-linked graphs [37]. To state this result we introduce several concepts. A disk representation of a graph $G$ is a drawing of $G$ in a disk in which no two edges cross. A 3-planar graph $(G, \mathcal{A})$ consists of a graph $G$ and a set $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

(i) for $i \neq j$, $N(A_i) \cap A_j = \emptyset$,

(ii) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and

(iii) if $p(G, \mathcal{A})$ denotes the graph obtained from $G$ by (for each $i$) deleting $A_i$ and adding edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in the plane with no edge crossings.

If, in addition, $b_0, b_1, \ldots, b_n$ are vertices in $G$ such that $b_i \notin A$ for $0 \leq i \leq n$ and $A \in \mathcal{A}$, $p(G, \mathcal{A})$ can be drawn in a closed disk with no edge crossings, and $b_0, b_1, \ldots, b_n$ occur on the boundary of the disk in this cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, ..., b_n)$ is 3-planar. If there is no need to specify $\mathcal{A}$, we may simply say that $(G, b_0, b_1, ..., b_n)$ is 3-planar. If $\mathcal{A} = \emptyset$, we say that $(G, b_0, b_1, ..., b_n)$ is planar. Moreover, we say that a face of (the disk representation of) $G$ is finite, if the face is inside the disk.

**Lemma 2.0.1 (Seymour, 1980)** Let $G$ be a graph with distinct vertices $x_1, x_2, x_3, x_4$. Then either $(G, x_1, x_2, x_3, x_4)$ is 3-planar, or $G$ has a cluster $\{X_1, X_2\}$ such that $\{x_1, x_3\} \subseteq X_1$
and \( \{x_2, x_4\} \subseteq X_2 \).

We say a sequence \((\alpha_1, \ldots, \alpha_n)\) is larger than \((\beta_1, \ldots, \beta_m)\) with respect to the lexicographic ordering if either

(i) \(m < n\) and \(\alpha_i = \beta_i\) for \(i = 1, \ldots, m\), or

(ii) there exists \(j\) with \(1 \leq j \leq \min(m, n)\) so that \(\alpha_j > \beta_j\) and \(\alpha_i = \beta_i\) for \(i = 1, \ldots, j - 1\).

We will also use the following lemma to modify a certain path.

**Lemma 2.0.2** Let \(G\) be a connected graph and \(P\) be a path in \(G\) between vertices \(u_1\) and \(u_2\) of \(G\), and let \(C\) denote a component of \(G - V(P)\). Then one of the following holds:

- \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 2\), \(V(P \cup C) \subseteq V(G_1)\), and \(|V(G_2 - G_1)| \geq 1\), or

- \(G\) has an induced path \(Q\) from \(u_1\) to \(u_2\) such that \(G - V(Q)\) is connected with \(C \subseteq (G - V(Q))\).

**Proof.** We choose a path \(Q\) in \(G\) from \(u_1\) to \(u_2\) and label the components of \(G - Q\) as \(C_1, \ldots, C_n\) such that \(C \subseteq C_1\) and \(|V(C_2)| \geq \cdots \geq |V(C_n)|\), and, subject to this, \(s(Q) := (|V(C_1)|, |V(C_2)|, \ldots, |V(C_n)|)\) is maximum under the lexicographical ordering. Note that \(Q\) is well defined because of \(P\).

Then \(Q\) is an induced path in \(G\). For, otherwise, let \(Q'\) be the induced path in \(G[Q]\) from \(u_1\) to \(u_2\) then \(s(Q') > s(Q)\), a contradiction. If \(n = 1\) then the assertion of the lemma holds. So assume \(n \geq 2\).

Let \(l_n, r_n \in N(C_n) \cap V(Q)\) such that \(Q[l_n, r_n]\) is maximal. We may assume there exists \(C_j\) with \(j < n\) such that \(N(C_j) \cap P(l_n, r_n) \neq \emptyset\); otherwise, \(G\) has a separation \((G_1, G_2)\) such that \(V(G_1 \cap G_2) = \{l_n, r_n\}\), \(V(P \cup C) \subseteq V(G_1)\), and \(V(C_n) \subseteq V(G_2)\), a contradiction.
Now let \( Q' \) be an induced path between \( u_1 \) and \( u_2 \) in \( G[Q \cup C_n] \) such that \( Q' \cap Q(l_n, r_n) = \emptyset \). Clearly, \( s(Q') > s(Q) \) under the lexicographical ordering, a contradiction. \( \square \)

In the remainder of this chapter, we will always assume that \( \gamma := (G, a_0, a_1, a_2, b_1, b_2) \) is a given rooted graph such that \( b_1 b_2 \notin E(G) \), \( a_i b_j \notin E(G) \) for \( i = 0, 1, 2 \) and \( j = 1, 2 \), and \( G^* := G + b_1 b_2 + \{ a_i b_j : i = 0, 1, 2 \) and \( j = 1, 2 \} \) is 6-connected. When we write \( a_{i+j} \), we understand that the subscript \( i + j \) is taken modulo 3. In the next two lemmas, we show that \( G \) does not certain separations.

**Lemma 2.0.3** \( G \) has no separation \((G_1, G_2)\) such that \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5, c_6\} \), \( |V(G_2 - G_1)| \geq 2 \), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1) \), and \((G_2, c_1, c_2, c_3, c_4, c_5, c_6) \) is planar.

**Proof.** For, otherwise, let \( G_2' := G_2 + \{c_1 c_2, c_2 c_3, c_3 c_4, c_4 c_5, c_5 c_6, c_6 c_1, c_1 c_3, c_3 c_5, c_5 c_1\} \), which is planar as \((G_2, c_1, c_2, c_3, c_4, c_5, c_6) \) is planar.

Since \( G^* \) is 6-connected, \( G_2 \) has at least one edge from each \( c_i \) to \( V(G_2 - G_1) \) and, hence, the number of edges in \( G_2 \) with at least one end in \( V(G_2 - G_1) \) is at least \((6|V(G_2 - G_1)| + 6)/2 = 3|V(G_2 - G_1)| + 3 = 3|V(G_2)| - 15 \). Thus, \( G_2' \) has at least \( 3|V(G_2)| - 15 + 9 = 3|V(G_2)| - 6 \) edges.

Thus, \( G_2' \) is a planar graph with exactly \( 3|V(G_2')| - 6 \) edges and each \( c_i \) has a unique neighbor in \( G_2 - G_1 \). Note that \( G_2' \) must be a planar triangulation. Therefore, the neighbors of \( c_1, \cdots, c_6 \) in \( G_2 - G_1 \) are the same. Hence, since \( G^* \) is 6-connected, \( |V(G_2 - G_1)| = 1 \), a contradiction. \( \square \)

**Lemma 2.0.4** \( G \) has no separation \((G_1, G_2)\) such that \( |V(G_1 \cap G_2)| = 4 \) and for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1) \), \( |V(G_2 - G_1)| \geq 4 \), \( a_{\pi(2)}, b_{3-j} \in V(G_1) \), and \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, V(G_1 \cap G_2)) \) is planar.

**Proof.** Suppose to the contrary that such a separation \((G_1, G_2)\) exists in \( G \) and let \( V(G_1 \cap G_2) = \{c_1, c_2, c_4\} \) such that \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_2, c_1) \) is planar. Let \( X := V(G_2 - G_1) - \{a_{\pi(0)}, a_{\pi(1)}, b_j\} \). Since \( G^* \) is 6-connected, we see that \( G_2 \) has at least two edges from \( b_j \) to \( X \) and at least three edges from \( a_{\pi(i)} \) to \( X \) for \( i \in [2] \).
Further, for any \( i \in [4] \), \( c_i \) has a neighbor in \( X \). For, otherwise, suppose, for some \( i \in [4] \), \( c_i \) has no neighbor in \( X \). Then by applying Lemma 2.0.3 to the separation \((G[V(G_1)] \cup \{c_i\}, G_2 - c_i)\) in \( G \), we see that \(|X| = 1\). It then follows from planarity that \( b_j \) has at most one neighbor in \( X \), a contradiction.

Hence, the number of edges in \( G_2 \) with at least one end in \( X \) is at least \((6|X|+1+1+1+3+3+2)/2 = 3|X|+6\). So \( G'_2 := G_2 + \{c_1c_2, c_2c_3, c_3c_4, c_4a_{\pi(1)}, a_{\pi(1)}b_j, b_ja_{\pi(0)}, a_{\pi(0)}c_1, c_2a_{\pi(0)}, c_2b_j, c_2c_4, c_3b_j\} \) has edges at least \(3|X| + 6 + 11 = 3(|X| + 7) - 4\). On the other hand, since \( G'_2 \) is planar (as \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_2, c_1)\) is planar), \( G'_2 \) has at most \(3(|X| + 7) - 6 \) edges, a contradiction. \( \square \)

For \( i \in \{0, 1, 2\} \), an \( a_i\)-frame in \( \gamma \) consists of disjoint paths \( A, B \) in \( G - a_i \) from \( a_{i-1}, b_1 \) to \( a_{i+1}, b_2 \), respectively, such that \( A \) is induced in \( G \), \( G - V(A) \) is connected, and the \( B \)-bridge of \( G \) containing \( a_i \) does not contain \( A \). The next lemma says that if \( \gamma \) is infeasible then it has a frame.

**Lemma 2.0.5** If \( \gamma \) is infeasible then there exists \( i \in \{0, 1, 2\} \) such that \( \gamma \) has an \( a_i\)-frame.

*Proof*. Since \( G^* \) is 6-connected, \( G - \{a_0, a_1, a_2\} \) contains an induced path \( P \) from \( b_1 \) to \( b_2 \) such that \( G - \{a_0, a_1, a_2\} - V(P) \neq \emptyset \). Now, by Lemma 2.0.2, \( G - \{a_0, a_1, a_2\} \) has an induced path \( Q \) from \( b_1 \) to \( b_2 \) such that \( C := G - \{a_0, a_1, a_2\} - V(Q) \) is connected and \( C \neq \emptyset \).

Note that there exists a permutation \( i, j, k \) of \( \{0, 1, 2\} \) such that \( N_{G^*}(a_j) \cap V(C) \neq \emptyset \) and \( N_{G^*}(a_k) \cap V(C) \neq \emptyset \), or \( N_{G^*}(a_j) \cap V(C) = \emptyset \) and \( N_{G^*}(a_k) \cap V(C) = \emptyset \). In the former case, \( G - a_i \) contains disjoint paths from \( b_1, a_j \) to \( b_2, a_k \), respectively. In the latter case, \( N_{G^*}(a_j) \cap V(Q(b_1, b_2)) \neq \emptyset \) and \( N_{G^*}(a_k) \cap V(Q(b_1, b_2)) \neq \emptyset \); so we have a path in \( G[Q(b_1, b_2) + \{a_j, a_k\}] \) from \( a_j \) to \( a_k \) and a path from \( b_1 \) to \( b_2 \) in \( G - \{a_0, a_1, a_2\} - V(Q(b_1, b_2)) \).

Hence, there exists \( i \in \{0, 1, 2\} \) such that \( G - a_i \) has disjoint paths \( A^* \) and \( B \) from \( a_{i-1}, b_1 \) to \( a_{i+1}, b_2 \), respectively. Since \( \gamma \) is infeasible, \( a_i \) and \( A^* \) are contained in different
components of $G - B$. Hence, $a_i$ and $B$ are contained in a component of $G - V(A^*)$.

So by Lemma 2.0.2, $G$ has an induced path $A$ between $a_{i-1}$ and $a_{i+1}$ such that $G - V(A)$ is connected and $V(B) \cup \{a_i\} \subseteq V(G - A)$. Since $\gamma$ is infeasible, the $B$-bridge of $G$ containing $a_i$ does not contain $A$. Hence, $A, B$ is an $a_i$-frame in $\gamma$. \hfill \Box

In the next two lemmas, we derive useful information about frames in $\gamma$.

**Lemma 2.0.6** Suppose $\gamma$ is infeasible and $A, B$ is an $a_i$-frame in $\gamma$. Let $A_i(B)$ denote the $B$-bridge of $G$ containing $a_i$, and let $V(A_i(B) \cap B) = \{d_1, \ldots, d_t\}$ such that $b_1, b_1, \ldots, d_t, b_2$ occur on $B$ in this order. Then $(A_i(B) \cup B, a_i, b_1, d_1, \ldots, d_t, b_2)$ is planar.

**Proof.** Let $G' = G/A$, and let $a'$ denote the vertex representing the contraction of $A$. Since $\gamma$ is infeasible, $G'$ has no disjoint paths from $a', b_1$ to $a_0, b_2$, respectively. So by Lemma 2.0.1, there exists a set $S$ of pairwise disjoint subsets of $V(G')$, such that $(G', S, a', b_1, a_i, b_2)$ is 3-planar.

Note that for any $S \in S$, $a' \in N_{G'}(S)$. For, otherwise, $N_G(S)$ is a cut in $G^*$ separating $S$ from $\{a_0, a_1, a_2, b_1, b_2\}$. But this contradicts the assumption that $G^*$ is 6-connected.

Thus, for any $S \in S$, we have $|N_{G'}(S) \cap V(B)| \leq 2$. Hence, $S \cap A_i(B) = \emptyset$. For otherwise, since $a' \in N_{G'}(S)$, there exists $u \in V(A_i(B) \cap B)$, such that $u \in S$. But then $G - V(A)$ contains three independent paths from $u$ to $b_1, b_2, a_i$, respectively, a contradiction to the existence of cut $N_{G'}(S)$. Therefore, $A_i(B) \subseteq G' - \cup S \subseteq S$, and $G' - \cup S \subseteq S$ has a disk representation with $b_1, b_2, a_i$ on the boundary of the disk. Thus, $A_i(B) \cup B$ inherits a disk representation with $b_1, b_2, a_i$ occurring on the boundary of the disk. Since $A_i(B) \cup B - B$ has only one component, $(A_i(B) \cup B, a_i, b_1, d_1, \ldots, d_t, b_2)$ is planar. \hfill \Box

Suppose $A, B$ is an $a_i$-frame in $\gamma$. Let $A_i(B)$ denote the $B$-bridge of $G$ containing $a_i$.

By a *doublecross* in $A, B$ we mean a pair of disjoint connected subgraphs $A', B'$ (in this order) of $G - (V(A_i(B)) \setminus V(B))$ for which there exist $a'_1, a'_2 \in V(A)$ and $b'_1, b'_2 \in V(B)$, such that $V(A')$ includes $a'_1, a'_2$ and at least one vertex of $B(b'_1, b'_2)$ and is otherwise disjoint from $A \cup B[b'_1, b'_2] \cup B[b'_2, b_2]$, and $V(B')$ includes $b'_1, b'_2$ and at least one vertex of $A(a'_1, a'_2)$.
and is otherwise disjoint from \( B \cup A[a_1, a'_1] \cup A[a'_2, a_2]. \) The vertices \( a'_1, a'_2, b'_2, b'_1 \) (in this order) are called the terminals of the doublecross.

**Lemma 2.0.7** If \( \gamma \) is infeasible then there is no double cross in any frame in \( \gamma. \)

**Proof.** Without loss of generality, assume \( A, B \) is an \( a_0 \)-frame in \( \gamma. \) Suppose \( A', B' \) is a double cross in \( A, B \) with terminals \( a'_1, a'_2, b'_2, b'_1. \) Let \( H = A(a'_1, a'_2) \cup B(b'_1, b'_2) \cup (A' - \{a'_1, a'_2\}) \cup (B' - \{b'_1, b'_2\}). \) Consider the graph \( G' \) obtained from \( G \) by contracting \( H \) to a single vertex \( h. \)

Since \( G' \) is 6-connected, then, combined with the existence of four disjoint paths \( A[a_1, a'_1], A[a'_2, a_2], B[b'_1, b'_1], B[b'_2, b'_2] \) and Menger’s theorem, \( G' \) contains five vertex-disjoint paths between \( \{a'_1, a'_2, b'_1, b'_2, h\} \) and \( \{a_0, a_1, a_2, b_1, b_2\}. \) So \( G \) contains five disjoint paths \( P_i, i = 1, \ldots, 5, \) (also internally disjoint from \( H \) joining \( a'_1, a'_2, b'_1, b'_2 \) and \( H \) to \( \{a_0, a_1, a_2, b_1, b_2\}. \)) Without loss of generality, assume that \( a_1 \in V(P_1), a_2 \in V(P_2), b_1 \in V(P_3), b_2 \in V(P_4), \) and \( a_0 \in V(P_5). \)

Let \( S_1 = (V(P_1 \cup P_2 \cup P_5)) \cap (\{a'_1, a'_2, b'_1, b'_2\} \cup V(H)), \) and \( S_2 = (V(P_3 \cup P_4)) \cap (\{a'_1, a'_2, b'_1, b'_2\} \cup V(H)). \) Using the properties of a double cross, we can show that \( H \) contains a cluster \( \{H_1, H_2\} \) such that \( S_i \subseteq V(H_i), i = 1, 2. \) Let \( X_1 := H_1 \cup V(P_1 \cup P_2 \cup P_3) \) and \( X_2 := V(P_3 \cup P_4) \cup H_2. \) Then \( \{X_1, X_2\} \) is a cluster in \( G, \) a contradiction. \( \qed \)

In the next two lemmas, we consider the intersection of special cuts in a planar graph, which may force another cut or interesting structures of the graph.

**Lemma 2.0.8** Let \( \gamma \) be infeasible with an \( a_0 \)-frame \( A, B, \) and let \( G_0 \) be obtained from \( G^* \) by deleting the component of \( G^* - B \) containing \( A. \) Suppose \( (G_0, a_0, b_1, B, b_2) \) is planar, and \( G_0 \) has 3-cuts \( \{a'_0, b'_1, b'_2\} \) and \( \{a''_0, b''_1, b''_2\} \) separating \( \{a_0, b_1, b_2\} \) from \( B[b'_1, b'_2] \) and \( B[b''_1, b''_2], \) respectively, such that \( b_1, b'_1, b''_1, b'_2, b''_2, b_2 \) occur on \( B \) in order, \( b'_1 \neq b''_2, \) and \( G_0 \) contains a path from \( B(b'_1, b''_2) \) to \( a_0, \) internally disjoint from \( B. \) Then one of the following holds:

(i) \( \{b'_1, b''_2\} \) is contained in a 3-cut of \( G_0 \) separating \( \{a_0, b_1, b_2\} \) from \( B[b'_1, b''_2]. \)
(ii) \( \{b''_1, b''_2\} = \{b_1, b_2\} \), and \( a'_0 = a''_0 = a_0 \).

(iii) \( \{a''_0, b''_1, b''_2\} = \{a_0, b_1, b_2\} \), \( b''_2 \) is a cut vertex of \( G_0 \) separating \( b_2 \) from \( \{a_0, b_1\} \), and \( a'_0, a''_0, b'_2, b''_2 \) are incident with a common finite face of \( G_0 \).

(iv) \( \{a'_0, b''_1, b''_2\} = \{a_0, b_1, b_2\} \), \( b'_1 \) is a cut vertex of \( G_0 \) separating \( b_1 \) from \( \{a_0, b_2\} \), and \( a'_0, a''_0, b'_1, b''_1 \) are incident with a common finite face of \( G_0 \).

**Proof.** We may assume \( a'_0 \neq a''_0 \). For, otherwise, since \( (G_0, a_0, b_1, B, b_2) \) is planar, either \( \{a'_0, b'_1, b'_2\} \) is a 3-cut in \( G_0 \) separating \( \{a_0, b_1, b_2\} \) from \( B[b''_1, b''_2] \) and (i) holds, or \( \{a'_0, b'_1, b'_2\} = \{a_0, b_1, b_2\} \) and (ii) holds.

For \( i \in [2] \), let \( F'_i \) be a finite face of \( G_0 \) incident with both \( b'_i \) and \( a'_0 \) and let \( F''_i \) be a finite face of \( G_0 \) incident with both \( b''_i \) and \( a''_0 \). Since \( a'_0 \neq a''_0, b'_1, b'_2, b''_1, b''_2 \) occur on \( B \) in order, and \( G_0 \) contains a path from \( B(b''_1, b''_2) \) to \( a_0 \) and internally disjoint from \( B \), we have \( F'_i = F''_i \) for some \( i \in [2] \).

By symmetry, we may assume \( F'_1 = F''_1 \). Then \( a'_0, a''_0, b'_1, b''_1 \) are incident with a common finite face of \( G_0 \). Thus, either \( \{a'_0, b''_1, b'_2\} \) is a 3-cut of \( G_0 \) separating \( \{a_0, b_1, b_2\} \) from \( B[b''_1, b''_2] \), or \( \{a'_0, b''_1, b'_2\} = \{a_0, b_1, b_2\} \) and \( b'_1 \) is a cut vertex of \( G_0 \) separating \( b_1 \) from \( \{a_0, b_2\} \). So (i) or (iv) holds, a contradiction.

**Lemma 2.0.9** Let \( \gamma \) be infeasible and \( A, B \) be an \( a_0 \)-frame in \( \gamma \), and let \( G_0 \) be obtained from \( G^* \) by deleting the component of \( G^* − B \) containing \( A \). Suppose \( (G_0, a_0, b_1, B, b_2) \) is planar, and \( G_0 \) has four distinct vertices \( b''_1, b'_1, b''_2, b'_2 \) with \( b_1, b''_1, b'_1, b''_2, b'_2, b_2 \) on \( B \) in order, and \( b''_1, b''_2 \) are incident with a common finite face of \( G_0 \).

(i) If \( \{b'_1, b'_2\} \) is a 2-cut in \( G_0 \) separating \( B[b'_1, b'_2] \) from \( \{a_0, b_1, b_2\} \), then \( b''_1, b'_1, b''_2, b'_2 \) are incident with a common finite face of \( G_0 \), and \( \{b''_1, b''_2\} \) is a 2-cut in \( G_0 \) separating \( B[b''_1, b''_2] \) from \( \{a_0, b_1, b_2\} \).

(ii) If there exists a vertex \( a'_0 \) in \( G_0 \), such that \( \{a'_0, b'_1, b'_2\} \) is a 3-cut in \( G_0 \) separating \( B[b'_1, b'_2] \) from \( \{a_0, b_1, b_2\} \), then one of the following occurs:
(a) \(a'_0, b''_1, b'_1, b''_2\) are incident with a common finite face of \(G_0\), and \(\{a'_0, b''_1, b'_2\}\) is a 3-cut in \(G_0\) separating \(B[b''_1, b'_2]\) from \(\{a_0, b_1, b_2\}\) or \(\{a'_0, b''_1, b'_2\} = \{a_0, b_1, b_2\}\).

(b) \(a'_0, b''_1, b'_2, b''_2\) are incident with a common finite face of \(G_0\), and \(\{b''_1, b'_2\}\) is a 2-cut in \(G_0\) separating \(B[b''_1, b'_2]\) from \(\{a_0, b_1, b_2\}\).

Proof. Let \(F''\) be a finite face of \(G_0\) incident with \(b''_1, b''_2\). To prove (i), we let \(F'\) be a finite face of \(G_0\) incident with \(b'_1, b'_2\). Since \(b'_1, b''_1, b'_1, b''_2, b'_2, b'_2\) occur on \(B\) in order, \(F' = F''\), and so (i) holds.

Next, we prove (ii). For each \(i \in [2]\), we let \(F'_i\) be a finite face of \(G_0\) incident with both \(b'_i\) and \(a'_0\). Since \(b_1, b''_1, b'_1, b''_2, b'_2, b_2\) occur on \(B\) in order, then \(F'_1 = F''\) or \(F'_2 = F''\). Now, if \(F'_1 = F''\), then (a) of (ii) holds; if \(F'_2 = F''\), then (b) of (ii) holds. \(\square\)
In this chapter, we show that if $\gamma$ is infeasible then $\gamma$ has a special frame. For an $a_i$-frame $A, B$ in $\gamma$, we fix the following notation:

- $\alpha(A, B) = |\{b_i : N(b_i) \cap V(A_i(B) - a_i - B) \neq \emptyset\}|$, and
- $c(A, B) = |\{v \in V(A_i(B) \cap B) - \{b_1, b_2\} : \{v, a_i\} \text{ separates } b_1 \text{ from } b_2 \text{ in } A_i(B) \cup B\}|$.

We say that an $a_i$-frame $A, B$ in $\gamma$ is good, if among all the frames in $\gamma$,

(i) $\alpha(A, B)$ is maximum,

(ii) subject to (i), $c(A, B)$ is minimum,

(iii) subject to (ii), $A_i(B)$ is maximal.

**Lemma 3.0.1** Suppose $\gamma$ is infeasible and $A, B$ is a good frame in $\gamma$. Let $i \in \{0, 1, 2\}$ and $A', B'$ be disjoint paths in $G - a_i$ from $a_{i-1}, b_1$ to $a_{i+1}, b_2$, respectively.

(i) If, for some $j \in [2]$, $G$ has a path $B_0$ from $a_i$ to $b_j$ and independent from $A', B'$, then

$\alpha(A, B) \geq 1$.

(ii) If $\{a_i, b_1, b_2\}$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, then

$\alpha(A, B) = 2$.

(iii) If $G$ has a path $B''$ from $b_1$ to $b_2$ and independent from $A', B'$, then

$\alpha(A, B) = 2$ and $c(A, B) = 0$.

**Proof.** We first prove (i). We see that $B', B_0$ are contained in a common component of $G - V(A')$. By Lemma 2.0.2 and the existence of $A'$, there exists an induced path $A^*$ from
$a_{i-1}$ to $a_{i+1}$, such that $G - V(A^*)$ is connected, and $B', B_0 \subseteq G - V(A^*)$. Since $\gamma$ is infeasible, $A^*$ and $a_i$ are in different components of $G - B'$. So $A^*, B'$ is a frame. By the existence of $B_0$, $\alpha(A^*, B') \geq 1$, and so $\alpha(A, B) \geq 1$.

Similarly, for (ii), let $C$ be the component of $G - (A' \cup (B' - \{b_1, b_2\}))$ containing $b_1, b_2, a_i$, we may assume there exists an induced path $A^*$ from $a_{i-1}$ to $a_{i+1}$, such that $G - V(A^*)$ is connected, and $B', C \subseteq G - V(A^*)$. So $A^*, B'$ is a frame. By the existence of $C$, $\alpha(A^*, B') = 2$, and so $\alpha(A, B) = 2$.

For (iii), since $\gamma$ is infeasible, $B' \cup B'' + a_i$ must be contained in a component of $G - V(A')$. Hence, we may assume that $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$. So by (ii), $\alpha(A, B) = 2$. Now by Lemma 2.0.2 and the existence of $A'$, there exists an induced path $A^*$ from $a_{i-1}$ to $a_{i+1}$, such that $G - V(A^*)$ is connected, and $B' \cup B'' + a_i \subseteq (G - V(A^*))$. So $A^*, B'$ is a frame. Since $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, we see that $c(A, B) = 0$. \hfill \Box

For a frame $A, B$ in $\gamma$, an $A$-$B$ bridge is an $(A \cup B)$-bridge of $G$ with at least three vertices and intersecting both $A$ and $B$. Let $M$ be an $A$-$B$ bridge, $l, r \in V(A \cap M)$, and $l', r' \in V(B \cap M)$, such that $A[l, r]$ and $B[l', r']$ are maximal. Then we say that $l, r$ are the extreme hands of $M$, and that $l', r'$ are the feet of $M$. We say that $M$ lies on $B[\overline{b'_1, b'_2}]$ for some $b'_1, b'_2 \in V(B)$, if $B[l', r'] \subseteq B[\overline{b'_1, b'_2}]$. We say that $M$ is fat if $|V(M \cap B)| \geq 2$ and non-fat if it is not fat.

**Lemma 3.0.2** Suppose $\gamma$ is infeasible and $A, B$ is a good $a_0$-frame in $\gamma$. Let $\{d_1, \cdots, d_t\} = V(B \cap A_0(B))$ such that $b_1, d_1, \cdots, d_t, b_2$ occur on $B$ in order, and let $d_0 = b_1, d_{t+1} = b_2$.

Then the following conclusions hold:

(i) For any $i \in [t]$, $G - (A_0(B) - (B - d_i))$ does not contain disjoint paths from $a_1, b_1$ to $a_2, b_2$, respectively.

(ii) For any $A$-$B$ bridge $M$, $M \cap B \subseteq B[d_{i-1}, d_i]$ for some $i \in [t + 1]$. 

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(iii) Let \( N \) be a \( B \)-bridge of \( G \) not containing \( A \) or \( a_0 \), then \(|V(N \cap B)| \geq 4\), and 
\[ N \cap B \subseteq B[d_{i-1}, d_i] \text{ for some } i \in [t+1]. \]

\[ \text{Proof.} \] First, we note that (ii) and (iii) follow immediately from (i). So we prove (i). Suppose (i) fails, and let \( A^*, B' \) be disjoint paths in \( G - (A_0(B) - (B - d_i))) \) from \( a_1, b_1 \) to \( a_2, b_2 \), respectively.

Then \( A_0(B) \cup B' \) is contained in a component of \( G - V(A^*) \). By Lemma 2.0.2 and the existence of \( A^* \), there exists an induced path \( A' \) from \( a_1 \) to \( a_2 \), such that \( G - V(A') \) is connected, and \( A_0(B) \cup B' \subseteq (G - V(A')) \). So \( A', B' \) is a frame in \( \gamma \). Now, due to the existence of \( d_i \), the \( B \)-bridge of \( G \) containing \( a_0 \) is properly contained in the \( B' \)-bridge of \( G \) containing \( a_0 \), a contradiction. \[ \square \]

An \( a_i \)-frame \( A, B \) in \( \gamma \) is \textit{ideal} if \( A, B \) is a good such that

\begin{enumerate}
\item the union of those \( B \)-bridges of \( G \) not containing \( A \) or \( a_i \) is maximal,
\item subject to (i), the union of those fat \( A-B \) bridges is maximal,
\item subject to (ii), the number of non-fat \( A-B \) bridges is minimum.
\end{enumerate}

\textbf{Lemma 3.0.3} Suppose \( \gamma \) is infeasible with ideal frame \( A, B \). Then all \( A-B \) bridges are fat.

\[ \text{Proof.} \] Let \( M \) be a non-fat \( A-B \) bridge with extreme hands \( l, r \) and foot \( u \). Then \( V(M \cap A(l, r)) \neq \emptyset \), to avoid the cut \( \{l, r, u\} \) in \( G^* \). Note that \( M - u - A(l, r) \) has a path from \( l \) to \( r \). Hence, by Lemma 2.0.2, \( M \cup A[l, r] - u \) contains an induced path \( P \) from \( l \) to \( r \), such that \( M \cup A[l, r] - u - V(P) \) is connected with \( A(l, r) \subseteq M \cup A[l, r] - u - V(P) \). Let \( A' := A[a_1, l] \cup P \cup A[r, a_2] \). We show that \( A', B \) contradicts the choice of \( A, B \).

Clearly, \( A', B \) is a good frame, and the union of those \( B \)-bridges of \( G \) not containing \( A \) or \( a_0 \) is equal to the union of those \( B \)-bridges of \( G \) not containing \( A' \) or \( a_0 \). Moreover, \( A(l, r) \) is contained in a non-fat \( A'-B \) bridge; otherwise, the union of those fat \( A'-B \) bridges properly contains the union of those fat \( A-B \) bridges, a contradiction.
Let $M_1, \ldots, M_k$ be the $A$-$B$ bridges such that for each $i \in [k]$, $M_i \cap A(l, r) \neq \emptyset$, $M_i \neq M$. Then $k \neq 0$; otherwise, $G$ has at least two disjoint edges from $A(l, r)$ to $B$ (as $G^*$ is 6-connected), which contradicts that $A(l, r)$ is contained in a non-fat $A'$-$B$ bridge.

Since $M_i \cap A(l, r) \neq \emptyset$ for $i \in [k]$, $\bigcup_{i \in [k]} M_i$ and $A(l, r)$ are contained in a same non-fat $A'$-$B$ bridge; so $M_1$, \ldots, $M_k$ are non-fat $A$-$B$ bridges. Now, since $M \cup A[l, r] - u - V(P)$ is connected with $A(l, r) \subseteq M \cup A[l, r] - u - V(P)$, then $\bigcup_{i \in [k]} M_i$ and $M \cup A[l, r] - u - V(P)$ are contained in one single $A'$-$B$ bridge. Hence, the number of non-fat $A'$-$B$ bridges is strictly smaller than the number of non-fat $A$-$B$ bridges, a contradiction. \hfill \Box

Let $A$, $B$ be a good $a_l$-frame in $\gamma$, let $\{d_1, \ldots, d_t\} = V(B \cap A_i(B))$ with $b_1, d_1, \ldots, d_t, b_2$ on $B$ in order, and let $d_0 = b_1$ and $d_{t+1} = b_2$. For any $i \in [t+1]$, we let $J_i^*$ be the union of $B[d_{i-1}, d_i]$, all the edges between $A$ and $B[d_{i-1}, d_i]$, all those $A$-$B$ bridges $M$ with $M \cap B \subseteq B[d_{i-1}, d_i]$, and all those $B$-bridges $N$ of $G$ with $(A + a_i) \cap N = \emptyset$ and $N \cap B \subseteq B[d_{i-1}, d_i]$. Let $u_1, u_2 \in V(A \cap J_i^*)$, such that $a_1, u_1, u_2, a_2$ occur on $A$ in order with $A[u_1, u_2]$ maximal. Then we say $J_i = G[V(J_i^* \cup A[u_1, u_2])]$ is an $A$-$B$ connector, and $u_1, u_2$ are the extreme hands of $J_i$. We say that $d_{i-1}, d_i$ are the feet of $J_i$. Note that our definition does not require $J_i \cap J_j = \emptyset$ for $i \neq j$.

An $A$-$B$ connector $J$ (with feet $v_1, v_2$ and extreme hands $u_1, u_2$) is slim if $(J - A[u_1, u_2], B[v_1, v_2])$ is planar, and each edge of $J$ with exactly one end in $A[u_1, u_2]$ has its other end in $B[v_1, v_2]$. Thus, no slim $A$-$B$ connector contains an $A$-$B$ bridge. If $J$ is not a slim connector, we say that $J$ is a fat $A$-$B$ connector.

**Lemma 3.0.4** Let $\gamma$ be infeasible with an ideal frame $A, B$. Let $J$ be an $A$-$B$ connector with feet $v_1, v_2$ and extreme hands $u_1, u_2$, such that $V(J) \setminus \{u_1, u_2, v_1, v_2\} \neq \emptyset$. Then

1. $u_1 \neq u_2$, there exists a unique $j \in [2]$ such that $G$ has an $A$-$B$ path from $B[v_j, v_j]$ to $A(u_1, u_2)$, and $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, and

2. if $J$ is fat then $N_G(v_j) \cap V(J - v_j - A) \not\subseteq L_p$ for $p \in [2]$, where $L_p$ denotes the subpath of the outer walk of $(J - v_j, A[u_1, u_2], v_{3-j})$ from $u_p$ to $v_{3-j}$ without going
through $u_{3-p}$.

Proof. Since $V(J) \setminus \{u_1, u_2, v_1, v_2\} \neq \emptyset$ and $G^*$ is 6-connected, then $u_1 \neq u_2$ and $G$ has an $A$-$B$ path from $B - B[b_1, b_2]$ to $A(u_1, u_2)$. By Lemma 2.0.7, there exists a unique $j \in [2]$ such that $G$ has an $A$-$B$ path from $B[b_j, v_j]$ to $A(u_1, u_2)$.

To prove $(J - v_j, A[u_1, u_2], v_3-j)$ is planar, let $T$ be an $A$-$B$ path from $t' \in B[b_j, v_j]$ to $t \in A(u_1, u_2)$. If $J - v_j$ contains disjoint paths $A^*, B^*$ from $u_1, t$ to $u_2, v_3-j$, respectively, then $A' := A[a_1, u_1] \cup A^* \cup A[u_2, a_2]$ and $B' := B[b_j, t'] \cup B^* \cup B[v_3-j, b_3-j]$ are disjoint paths in $G - v_j = V(A_0(B) - B)$ from $a_1, b_1$ to $a_2, b_2$, respectively; which contradicts (i) of Lemma 3.0.2. So assume that such $A^*, B^*$ do not exist. Then by Theorem 2.0.1, there exist $m \geq 0$ and a set $D = \{D_1, \ldots, D_m\}$ of pairwise disjoint nonempty subsets of $V(J - v_j) - \{u_1, u_2, t, v_3-j\}$ such that $(J - v_j, D, u_1, t, u_2, v_3-j)$ is 3-planar. We choose $D_1, \ldots, D_m$ such that $\bigcup_{i \in [m]} D_i$ is minimal. Then for all $p \in [m]$, $G[D_p \cup N_{J-v_j}(D_p)]$ does not have a disk representation with $N_{J-v_j}(D_p)$ occurring on the boundary of the disk (or else, $D_p$ could be chosen to be empty). Obviously, $|D_p| \geq 2$.

Note that $J - v_j - A[u_1, u_2]$ is connected. For, otherwise, let $C$ be a component of $J - v_j - A[u_1, u_2]$ disjoint from $B(v_j, v_{3-j})$. Then $N_G(C) \subseteq V(A[u_1, u_2]) \cup \{v_j\}$. Since $G - A$ is connected, $v_j \in N(C)$; hence, $G[V(C) \cup N(C)] - E(A)$ is a non-fat $A$-$B$ bridge, contradicting Lemma 3.0.3.

If $m = 0$ then $D = \emptyset$, and $(J - v_j, u_1, t, u_2, v_3-j)$ is planar; so $(J - v_j, A[u_1, u_2], v_3-j)$ is planar as $J - v_j - A[u_1, u_2]$ is connected. Hence, $m \geq 1$. Since $G^*$ is 6-connected, for all $p \in [m]$, $N_{J-v_j}(D_p) \cup \{v_j\}$ is not a cut of $G$ separating $D_p$ from other vertices. So $D_p \cap V(A) \neq \emptyset$. Since $D_p \cap \{u_1, u_2, t, v_3-j\} = \emptyset$, $|N_{J-v_j}(D_p) \cap A| \geq 2$. Moreover, since $A$ is an induced path and $G[D_p \cup N_{J-v_j}(D_p)]$ does not have a disk representation with $N_{J-v_j}(D_p)$ occurring on the boundary of the disk, $D_p \not\subseteq V(A)$. Thus, $N_{J-v_j}(D_p) \not\subseteq V(A)$ as $J - v_j - A[u_1, u_2]$ is connected. So $|N_{J-v_j}(D_p)| = 3$ and $|N_{J-v_j}(D_p) \cap A| = 2$. Moreover, if we let $\{s_1, s_2, s\} = N_{J-v_j}(D_p)$ such that $s \not\in V(A)$ and $u_1, s_1, s_2, u_2$ occur on $A$ in order, then $J - v_j$ has a path $D$ from $s$ to $v_{3-j}$ disjoint from $A$; or else, there exists...
a non-fat $A$-$B$ bridge with foot $v_j$, or $G - V(A)$ is not connected. Moreover, since $G'$ is 6-connected, $G$ has an $A$-$B$ path $R$ from $r' \in V(B - B[v_1, v_2])$ to $r \in V(A(s_1, s_2))$. By Lemma 2.0.7, $r' \in B[b_j, v_j]$. 

Let $H := G[D_p \cup N_{j'-v_j}(D_p)]$. If $H$ contains disjoint paths $X', R_1$ from $s_1, r$ to $s_2, s$, respectively, then the paths $A' := A[a_1, s_1] \cup X' \cup A[a_2, s_2]$ and $B' := B[b_j, r'] \cup R \cup R_1 \cup D \cup B[v_3-j, b_3-j]$ in $G - V(A_0(B) - B) - \{v_j\}$ from $a_1, b_1$ to $a_2, b_2$, respectively, contradict Lemma 3.0.2. So such $X'$ and $R_1$ do not exist. By Lemma 2.0.1, there exist $n \geq 0$ and a set $V = \{V_1, \cdots, V_n\}$ of pairwise disjoint subsets of $D_p$ such that $(H, V, s_1, r, s_2, s)$ is 3-planar. However, we see that $\{D_1, \cdots, D_m\} \setminus \{D_p\} \cup \{V_1, \cdots, V_n\}$ contradicts our choice of $\{D_1, \cdots, D_m\}$. This completes the proof of (i).

Next, we prove (ii). Since $J$ contains disjoint paths $A[u_1, u_2]$ and $B[v_1, v_2]$, $N_G(v_j) \cap V(J - v_j - A) \neq \emptyset$. Suppose $N_G(v_j) \cap V(J - v_j - A) \subseteq L_p$ for some $p \in [2]$. Let $u \in N_G[v_j] \cap V(L_p)$, such that $u \neq u_p$, and $L_p[u_p, u]$ is minimal. Since $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, $J - v_j - A[u_1, u_2]$ is also planar. Let $P'$ denote the subpath of the outer walk of $J - v_j - A[u_1, u_2]$ from $u$ to $v_{3-j}$ with $P' \subseteq L_p$. Then $N_G(v_j) \cap V(J - v_j - A) \subseteq V(P')$. Let $B' = B[b_j, v_j] \cup \{v_ju\} \cup P' \cup B[v_{3-j}, b_{3-j}]$. Then $A, B'$ is a good frame. The union of those $B$-bridges of $G$ not containing $A$ and $a_0$ is contained in the union of those $B'$-bridges of $G$ not containing $A$ and $a_0$, which forces $B = B'$ by the choice of $A, B$. Moreover, by Lemma 3.0.3 and the planarity of $J - v_j$, each edge of $J$ with exactly one end in $A[u_1, u_2]$ has its other end in $B[v_1, v_2]$; so $J$ is a slim connector, a contradiction. \[\Box\]
In this chapter, we consider the situation when there is a fat connector for some ideal frame in $\gamma$. The first two lemmas study the inside of fat connectors, and show that each fat connector has a core in which we can find various disjoint paths.

**Lemma 4.0.1** Suppose $\gamma$ is infeasible and $A, B$ is an ideal $a_0$-frame in $\gamma$. Let $J$ be a fat $A$-$B$ connector with feet $v_1, v_2$, and extreme hands $u_1, u_2$, such that $(J - v_1, A[u_1, u_2], v_2)$ is planar; $a_1, u_1, u_2, a_2$ occur on $A$ in order, $b_1, v_1, v_2, b_2$ occur on $B$ in order, and $G$ has an $A$-$B$ path from $A(u_1, u_2)$ to $B[b_1, v_1]$. Then there exists a separation $(H, L)$ in $J$ of order 4 (we allow $H = J$ and $L$ consists of $u_1, u_2, v_2$ and no edges), such that

(i) $V(H \cap L) = \{v_1, x_1, x_2, y_2\}$, $u_1, x_1, x_2, u_2$ occur on $A$ in order, $v_1, y_2, v_2$ occur on $B$ in order, $A[x_1, x_2] \cup B[v_1, y_2] \subseteq H, \{u_1, u_2, v_2\} \subseteq V(L);$ 

(ii) $(L - A, B[y_2, v_2], v_1)$ is planar, and each edge of $L$ with exactly one end in $A$ has its other end in $V(B[y_2, v_2]) \cup \{v_1\};$

(iii) $(H - v_1, A[x_1, x_2], y_2)$ is planar, $H - v_1 - A[x_1, x_2]$ is connected, $x_1y_2, x_2y_2 \notin E(H)$, $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ contains disjoint paths from $v_1, y_2$ to $x_1, x_2$, respectively, and disjoint paths from $v_1, y_2$ to $x_2, x_1$, respectively, and $V(X_1 \cap X_2) = \{y_2\}$ and $N(v_1) \cap V(H - A) \nsubseteq V(X_i)$ for $i \in [2]$, where $X_i$ is the path from $x_i$ to $y_2$ on the outer walk of $H - v_1$ without going through $x_{3-i}$.

**Proof.** Note that by Lemma 3.0.4, if we take $H = J$ and let $L$ consist of $u_1, u_2, v_2$ and no edges, then $(H, L)$ satisfies (i) and (ii) (with $x_i = u_i$ for $i \in [2]$ and $y_2 = v_2$). Hence, we choose $(H, L)$ satisfying (i) and (ii) and, subject to this, $H$ is minimal. We show that (iii) holds.

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Since \((J - v_1, A[u_1, u_2], v_2)\) is planar, \((H - v_1, A[x_1, x_2], y_2)\) is planar. Note that \(H - v_1 - A[x_1, x_2]\) is connected; for otherwise, let \(C\) be a component of \(H - v_1 - A[x_1, x_2]\) not containing \(y_2\), which is also a component of \(J - v_1 - A[u_1, u_2]\). Then either it contradicts the definition of frame that \(G - V(A)\) is connected, or it contradicts Lemma 3.0.3 that all \(A-B\) bridges are fat. By the minimality of \(H\), we see that \(x_1y_2, x_2y_2 \notin E(H)\).

For \(i = 1, 2\), let \(X_i\) denote the path in the outer walk of \(H - v_1\) from \(y_2\) to \(x_i\) not containing \(x_{3-i}\). Then \(V(X_1 \cap X_2) = \{y_2\}\). For, otherwise, \(H\) has a separation \((H_1, H_2)\) such that \(|V(H_1 \cap H_2)| = 1, y_2 \in V(H_1 - H_2)\), and \(A[x_1, x_2] \subseteq H_2\). Since \(G^{*}\) is 6-connected, \(V(H_1 - H_2) = \{y_2\}\). Let \(y'_2 \in V(H_1 - y_2)\). Now it is easy to check that the separation \((H - y_2, G[L + y'_2])\) contradicts the choice of \((H, L)\) (that \(H\) is minimal).

Next we show that \(N(v_1) \cap V(H - A) \not\subseteq V(X_i)\) for \(i = 1, 2\). For, suppose this is false and, by symmetry, that \(N(v_1) \cap V(H - A) \subseteq V(X_2)\). Let \(y'_2 \in N(v_1) \cap V(X_2)\) with \(X_2[y'_2, y_2]\) minimal. Let \(B'\) denote the path in the outer walk of \(H - A\) from \(y'_2\) to \(y_2\) not containing \(X_2[y'_2, y_2]\). We could choose \(B\) so that \(B' \subseteq B\). However, this shows that \(J\) is not fat, a contradiction.

It remains to show that for \(j \in [2]\), \(H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}\) contains disjoint paths from \(v_1, y_2\) to \(x_{3-j}, x_j\), respectively. For, otherwise, we may assume by symmetry that \(H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}\) does not have disjoint paths from \(v_1, y_2\) to \(x_1, x_2\), respectively. Hence, \(H - A(x_1, x_2) - X_2 - \{v_1x_1, v_1x_2\}\) has no path from \(v_1\) to \(x_1\). Since \((H - v_1, A[x_1, x_2], X_2, X_1)\) is planar, there exist \(x'_1 \in V(A(x_1, x_2))\), \(y'_2 \in V(X_2)\), and a 2-separation \((H_1, H_2)\) in \(H - v_1\) such that \(V(H_1 \cap H_2) = \{x'_1, y'_2\}\), \(x_1, y_2 \in V(H_1)\), \(A[x'_1, x_2] \subseteq H_2\), and \(N(v_1) \cap V(H) \subseteq V(H_2 \cup A[x_1, x_2] \cup X_2)\). Then we see that the separation \((H_2, G[H_1 \cup L])\) of \(J\) contradicts the choice of \((H, L)\).

With the notation in Lemma 4.0.1, we say that \(H\) is an \(A-B\) core or a core of the fat connector \(J\). Moreover, we say that \(x_1, x_2\) are the extreme hands of \(H\), \(v_1, y_2\) are the feet of \(H\), and \(y_2\) is the main foot of \(H\). For convenience, we write \(y_1 := v_1\). By symmetry, we may always assume that \(a_1, x_1, x_2, a_2\) occur on \(A\) in order, and that \(b_1, y_1, y_2, b_2\) occur on \(B\).
in order. Note that $y_1 \in V(A_0(B))$ and $G$ has a path from $a_0$ to $y_1$ internally disjoint from $B$. For $i \in [2]$, let $x'_i \in V(A(x_1, x_2))$ such that $x'_i, x_i$ are incident with a common finite face of $H - y_1$, and $H - y_1$ has a path from $x'_i$ to $y_2$ and internally disjoint from $A$. And for $i \in [2]$, let $X'_i$ be the path from $y_2$ to $x'_i$ on the outer walk of $H - \{y_1, x_i\}$ without going through $x_{3-i}$.

**Lemma 4.0.2** Suppose $\gamma$ is infeasible, $A, B$ is an ideal $a_0$-frame, and $H$ is an $A$-$B$ core with extreme hands $x_1, x_2$ and feet $y_1, y_2$, where $y_2$ is the main foot. Then the degree of $y_2$ in $H - y_1$ is at least 2 and, for $i \in [2]$, $|V(X_i(x_i, y_2))| \geq 1$ and $V(X_i \cap X'_{3-i}) = \{y_2\}$. Moreover, if, for some $i \in [2]$, $H$ does not contain disjoint paths from $y_1, y_2$ to $x_i, x'_{3-i}$, respectively, and internally disjoint from $A$, then the following are true:

(i) No finite face of $H - y_1$ is incident with both $y_2$ and a vertex of $A(x_1, x_2)$.

(ii) For any $v \in N(y_1) \cap V(H)$ with $v \notin X'_{3-i} \cup A(x_i, x_{3-i})$, there exist $c_1 \in A(x_i, x'_{3-i})$ and $c_2 \in X'_{3-i}(x'_{3-i}, y_2)$, such that $\{c_1, c_2\}$ is a cut in $H - \{y_1, x_{3-i}\}$ separating $v$ from $x_i$, and there exist independent paths from $v$ to $c_1, c_2$ in $H - \{y_1, x_{3-i}\}$, respectively, which are internally disjoint from $X'_{3-i} \cup A[x_i, x'_{3-i}]$.

(iii) $H$ has disjoint paths from $y_1, y_2$ to $x_{3-i}, x'_i$, respectively, and internally disjoint from $A$.

**Proof.** By Lemma 4.0.1, $V(X_i \cap X_2) = \{y_2\}$ and $x_1y_2, x_2y_2 \notin E(H)$; so the degree of $y_2$ in $H - y_1$ is at least 2 and $|V(X_i(x_i, y_2))| \geq 1$. Moreover, $V(X_i \cap X'_{3-i}) = \{y_2\}$ for $i \in [2]$; for, suppose there exists $c \in V(X_i \cap X'_{3-i}) - \{y_2\}$, then $\{c, y_1, y_2, x_{3-i}\}$ is a cut in $G$ separating $V(X_{3-i})$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

By symmetry, we may assume that $H$ does not contain disjoint paths from $y_1, y_2$ to $x_1, x'_i$, respectively, that are internally disjoint from $A$.

To prove (i), suppose there exists $v_0 \in V(A(x_1, x_2))$ such that $v_0, y_2$ are incident with a common finite face in $H - y_1$. Since $(H - y_1, A[x_1, x_2], y_2)$ is planar, $H - y_1$ has a
separation \((H_1, H_2)\) such that \(V(H_1 \cap H_2) = \{y_2, v_0\}\), \(X_1 \subseteq H_1\), and \(X_2 \subseteq H_2\). Now, we further choose \(v_0\) so that \(H_1\) is minimal.

Now, we see that \(H_2\) contains a path \(P_2\) from \(y_2\) to \(x_2'\) and internally disjoint from \(A\); for otherwise, \(V(H_2 \cap H) = \{x_2\}\) and, hence, \(\{y_1, y_2, x_2\}\) is a cut in \(G\) separating \(V(X_2)\) from \(\{a_0, a_1, a_2, b_1, b_2\}\), a contradiction.

Now, let \(P_1\) be the path from \(y_1\) to \(x_1\) in \(H - V(A(x_1, x_2))\) (by (iii) of Lemma 4.0.1). Since \(v_0 \neq x_1\), \(V(P_1 \cap H) = \emptyset\), and so \(V(P_1 \cap P_2) = \emptyset\). However, the existence of \(P_1, P_2\) contradicts that \(H\) does not contain disjoint paths from \(y_1, y_2\) to \(x_1, x_2'\), respectively, and internally disjoint from \(A\). This completes the proof of (i).

To prove (ii), let \(v \in N(y_1) \cap V(H)\) such that \(v \notin X_2' \cup A(x_1, x_2)\). Since \((H - \{y_1, x_2\}, A[x_1, x_2] \cup X_2'[x_2', y_2])\) is planar and \(H - y_1 - A(x_1, x_2) \cup X_2'\) does not have a path from \(v\) to \(x_1\), there exist \(c_1, c_2 \in V(A(x_1, x_2) \cup X_2')\) such that \(\{c_1, c_2\}\) is a cut in \(H - \{y_1, x_2\}\) separating \(v\) from \(x_1\). We may assume \(c_1, c_2\) occur on \(A(x_1, x_2) \cup X_2'[x_2', y_2]\) in order.

Note that \(c_1 \notin V(X_2')\), to avoid the cut \(\{c_1, c_2, y_1, x_2\}\) in \(G^*\). Moreover, \(c_2 \notin A(x_2', y_2)\); or else, \(H - V(A) \cup \{y_1\}\) is not connected, contradicting (iii) of Lemma 4.0.1.

We choose \(c_1, c_2\) such that \(A[c_1, x_2']\) and \(X_2'[x_2', c_2]\) are minimal. Then \(H - \{y_1, x_2\}\) contains independent paths from \(v\) to \(c_1, c_2\), respectively, and internally disjoint from \(A \cup X_2'\). Moreover, by (i), \(c_2 \neq y_2\). This completes the proof of (ii).

To prove (iii), observe that \(V(X_1' \cap X_2') = \{y_2\}\). For otherwise, let \(c \in V(X_1' \cap X_2')\) with \(c \neq y_2\). Since \(y_2\) has degree at least 2 in \(H - y_1\) and \(x_1y_2, x_2y_2 \notin E(H)\), \(\{x_1, x_2, y_1, y_2, c\}\) is a cut in \(G^*\) separating \(V(X_1 \cup X_2)\) from \(\{a_0, a_1, a_2, b_1, b_2\}\), a contradiction.

Now, let \(u_2 \in V(X_2 \cap X_2')\) such that \(X_2'[x_2, u_2]\) is minimal. Moreover, let \(v \in N(y_1) \cap V(H - A)\). If \(v \in V(X_2')\) then let \(P_2 = v = c_2\); and if \(v \notin V(X_2')\) then by (ii), there exist \(c_1 \in V(A(x_1, x_2))\) and \(c_2 \in V(X_2'(x_2', y_2))\), such that \(\{c_1, c_2\}\) is a cut in \(H - \{y_1, x_2\}\) separating \(v\) from \(x_1\), and there exists a path \(P_2\) from \(v\) to \(c_2\) in \(H - \{y_1, x_2\}\), which is internally disjoint from \(X_2' \cup A[x_1, x_2']\). Since \(V(X_1' \cap X_2') = \emptyset\) and \((H - y_1, A[x_1, x_2'] \cup X_2')\)
is planar, \(P_2\) is disjoint from \(X'_1\). Now, \(X'_1\) and \(y_1v \cup P_2 \cup X'_2[u_2, x_2] \) are disjoint paths from \(y_2, y_1\) to \(x'_1, x_2\), respectively, in \(H\), which are internally disjoint from \(A\).

\[\square\]

The next lemma describes some interactions between cores from different connectors and finds a path \(B'\) so that \(A, B'\) is a good frame in \(\gamma\) which will eventually be used to form a special frame \(A', B'\) in \(\gamma\).

Lemma 4.0.3 Let \(\gamma\) be infeasible with an ideal \(a_0\)-frame \(A, B\), and let \(H^j, j \in [m]\), be the \(A-B\) cores in \(\gamma\) such that \(H^j\) has extreme hands \(x^j_1, x^j_2\) and feet \(y^j_1, y^j_2\). Then

(i) for any distinct \(i, j \in [m]\), \(A[x^i_1, x^i_2] \subseteq A[x^j_1, x^j_2]\) or \(A[x^i_1, x^i_2] \subseteq A[x^j_1, x^j_2]\).

(ii) for any \(j \in [m]\), \(H^j - A[x_1, x_2]\) has a path \(P_j\) from \(y_1\) to \(y_2\) such that \(|V(P_j)| \geq 3\), \(H^j - P_j\) is connected, and \(P_j\) is induced in \(G - y_1y_2\).

(iii) \(A, B'\) is a good \(a_0\)-frame and \(A_0(B') = A_0(B)\), where \(B'\) is obtained from \(B\) by replacing \(B[y^j_1, y^j_2]\) with the path \(P_j\) in (ii) for \(j \in [m]\), and

(iv) with \(G'_0\) as the graph obtained from \(G\) by deleting the component of \(G - B'\) containing \(A, (G'_0, a_0, b_1, B', b_2)\) is planar and, for any \(v \in B'(y^j_1, y^j_2)\), the degree of \(v\) in \(G'_0\) is 2.

Proof. To prove (i), assume for some distinct \(i, j \in [m]\) with \(i \neq j\), we have \(A[x^i_1, x^i_2] \nsubseteq A[x^j_1, x^j_2]\), and \(A[x^i_1, x^i_2] \nsubseteq A[x^j_1, x^j_2]\). Without loss of generality, let \(b_1, y^i_1, y^i_2, y^j_1, y^j_2, b_2\) occur on \(B\) in this order, and \(a_1, x^i_1, x^i_2, a_2\) occur on \(A\) in this order with \(x^i_2, x^i_1 \in A(x^i_1, x^i_2)\). By Lemma 4.0.1, \(H^i - A(x^i_1, x^i_2)\) has two disjoint \(A-B\) paths \(P_1, P_2\) from \(y^i_1, y^i_2\) to \(x^i_2, x^i_1\), respectively, and \(H^j - A(x^j_1, x^j_2)\) has two disjoint \(A-B\) paths \(P_3, P_4\) from \(y^j_1, y^j_2\) to \(x^j_2, x^j_1\), respectively. Therefore, \(P_1, P_2, P_3, P_4\) form a doublecross in \(A, B\), a contradiction.

For (ii), let \(j \in [m]\). Since \(H^j\) is a core, \(H^j - y^j_1y^j_2 - A\) has a path \(T_j\) from \(y^j_1\) to \(y^j_2\). So by Lemma 2.0.2, \(H^j - y^j_1y^j_2\) has an induced path \(P_j\) from \(y^j_1\) to \(y^j_2\) such that \(H^j - y^j_1y^j_2 - V(P_j)\) is connected and \(A[x^j_1, x^j_2] \subseteq H^j - y^j_1y^j_2 - P_j\).
To see (iii), we observe that $A_0(B')$, the $B'$-bridge of $G$ containing $a_0$, is the same as, $A_0(B)$, the $B$-bridge of $G$ containing $a_0$. So $A, B'$ is also a good $a_0$-frame.

To prove (iv), let $C$ denote the component of $G - B'$ containing $A$; so $G_0' = G - C$. By Lemma 2.0.6, $(A_0(B'), a_0, b_1, B', b_2)$ is planar. Thus, to show that $(G_0', a_0, b_1, B', b_2)$ is planar, it suffices to show that for any $A-B$ connector $J$ with feet $v_1, v_2, (J - C, B'[v_1, v_2])$ is planar. This is clear when $J$ is a slim connector. So assume $J$ is a fat connector. Then $J$ has a separation $(H, L)$ satisfying (i), (ii), and (iii) of Lemma 4.0.1. By (ii) of Lemma 4.0.1, $(L - A, B' \cap L)$ is planar. Since $H - B' \subseteq C$, we see that $(J - C, B'[v_1, v_2])$ is planar.

Moreover, for any $v \in B'(y_1^j, y_2^j)$, since $B'[y_1^j, y_2^j]$ is a path in the core $H^j$, then, combined with (ii) that $P_j$ is induced in $G - y_1 y_2$, the degree of $v$ in $G_0'$ is exactly 2. \hfill \Box

In the remainder of this chapter, suppose $\gamma$ is infeasible and $A, B$ is an ideal frame in $\gamma$. By (i) of Lemma 4.0.3, there exists an $A-B$ core (or said an $A-B'$ core) $H$ with extreme hands $x_1, x_2$ and feet $y_1, y_2$ ($y_2$ as the main foot), such that for any core $H^j$ with extreme hands $x_1^j, x_2^j$, we have $A[x_1^j, x_2^j] \subseteq A[x_1, x_2]$. We call such a core $H$ a main $A-B'$ core (or said a main $A-B$ core). We also use $B'$ to denote the path in (iii) of Lemma 4.0.3 and $G_0'$ to denote the graph in (iv) of Lemma 4.0.3. By (iii) of Lemma 4.0.2, for $i \in [2]$, we let $P_{1,i}, P_{2,3-i}$ be disjoint paths in $H - A(x_1, x_2)$ from $x_1, x_2$ to $y_i, y_{3-i}$, respectively.

We consider the structure of $G$ outside $H$. Let $r_1 \in V(B'[b_1, y_1])$, such that $B'[b_1, r_1]$ contains no foot of $A-B$ cores in $\gamma$, $G$ has no $A-B'$ path from $A(x_1, x_2)$ to $B'[b_1, r_1]$, and subject to these conditions, $B'[b_1, r_1]$ is maximal. Then $G$ has a path $R_1$ from $r_1$ to some $r \in V(A(x_1, x_2))$ and internally disjoint from $A$ such that $R_1 = r_1 r$ or $R_1$ is contained in some $A-B$ core $H'$ with $r_1$ as a foot and does not contain the other foot of $H'$.

For notational convenience, we let $t_1 := r_1$ and $t_2 := y_2$. We derive useful structure of $G$ outside $A[x_1, x_2] \cup B'[t_1, t_2]$.

**Lemma 4.0.4** $G$ has no $A-B'$ path from $A(x_1, x_2)$ to $B' - B'[t_1, t_2]$ or from $B'(t_1, t_2)$ to $A - A[x_1, x_2]$.  

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Lemma 4.0.5 Let $e_3 = a_3b_3, e_4 = a_4b_4 \in E(G)$ with $a_3, a_4 \in V(A)$ and $b_3, b_4 \in V(B')$.

(i) If for some $i \in [2], a_3 \in V(A[a_i, x_i]), b_3 \in V(B'[t_2, b_2]), a_4 \in V(A(a_3, x_i)$, and $b_4 \in V(B'[b_1, t_1]),$ then $G_0'$ has a 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \in B'[b_1, b_4]$ and $b_2' \in B'[t_2, b_3]$, which separates $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$ in $G_0'$.

(ii) If for some $i \in [2], a_3 \in V(A[a_i, x_i]), b_3 \in V(B'(b_1, t_1)), a_4 \in V(A(a_3, x_i)$, and $b_4 \in V(B'(t_2, b_2)), then one of the following holds:

(a) $G_0'$ has a 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \in B'[b_3, t_1]$ and $b_2' \in B'[b_4, b_2]$, which separates $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$ in $G_0'$;

(b) $G_0'$ has a 2-cut $\{y_1, b_2'\}$ with $b_2' \in B'[b_4, b_2]$, which separates $B'[y_1, b_2']$ from $\{a_0, b_1, b_2\}$ in $G_0'$.

(iii) If $a_3 \in V(A[a_1, x_1]), a_4 \in V(A[x_2, a_2]), b_3, b_4 \in V(B'(b_1, t_1)), then G_0'$ has a 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \in B'[b_3, b_1]$ and $b_2' \in B'[t_2, b_2]$, which separates $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$ in $G_0'$.

(iv) If $a_3 \in V(A[a_1, x_1]), a_4 \in V(A[x_2, a_2]), b_3, b_4 \in V(B'(t_2, b_2)), then one of the following holds:

(a) $G_0'$ has a 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \in B'[b_1, t_1]$ and $b_2' \in B'[b_3, b_4]$, which separates $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$ in $G_0'$;

(b) $G_0'$ has a 2-cut $\{y_1, b_2'\}$ with $b_2' \in B'[b_3, b_4]$, which separates $B'[y_1, b_2']$ from $\{a_0, b_1, b_2\}$ in $G_0'$.

Proof. By the maximality of $B'[b_1, r_1], G$ has no $A-B'$ path from $A(x_1, x_2)$ to $B'[b_1, t_1]$.

Since no double cross exists in $A, B$ (by Lemma 2.0.7), $G$ has no $A-B'$ path from $A(x_1, x_2)$ to $B'(t_2, b_2)$. Moreover, $G$ has no $A-B'$ path from $B'(t_1, t_2)$ to $A[a_1, x_1]\cup A(x_2, a_2]$; to avoid forming a double cross with $R_1$ and one of $\{P_{1,2}, P_{2,1}\}, \{P_{1,1}, P_{2,2}\}$ in $A, B$. 

\qed
Proof. Suppose (i) fails. Then, since \((G'_0, a_0, b_1, B', b_2)\) is planar and \(y_2\) is the main foot of \(H\), there exist disjoint paths \(B'_2, A'_0\) in \(G'_0 - (B'[b_1, b_4] \cup B'[y_2, b_3])\) from \(b_2, a_0\) to \(y_1, r_1\), respectively. Now, \(A[a_i, a_3] \cup e_3 \cup B'[y_2, b_3] \cup P_{3-i,2} \cup A(x_i, a_3-i] \cup R_1 \cup A'_0\) and \(B'[b_1, b_4] \cup e_4 \cup A[a_4, x_i] \cup P_{i,1} \cup B'_2\) show that \(\gamma\) is feasible, a contradiction.

Now suppose (ii) fails. Then, since \((G'_0, a_0, b_1, B', b_2)\) is planar and \(y_2\) is the main foot of \(H\), \(G'_0 - (B'[b_3, b_4] \cup B'[t_2, b_2])\) contains two disjoint paths \(B'_1, A'_0\) from \(b_1, a_0\) to \(y_1, y_2\), respectively. Now \(A[a_i, a_3] \cup e_3 \cup B'[b_3, r_1] \cup R_1 \cup A(x_i, a_3-i] \cup P_{3-i,2} \cup A'_0\) and \(B'_1 \cup P_{i,1} \cup A[a_4, x_i] \cup e_4 \cup B'[b_4, b_2]\) show that \(\gamma\) is feasible, a contradiction.

If (iii) fails then, since \((G'_0, a_0, b_1, B', b_2)\) is planar and \(y_2\) is the main foot of \(H\), \(G'_0 - (B'[b_3, b_4] \cup B'[t_2, b_2])\) has disjoint paths \(B'_1, A'_0\) from \(b_1, a_0\) to \(r_1, y_1\), respectively. Moreover, by Lemma 4.0.2, for some \(p \in [2]\), \(H\) contains disjoint paths \(Y_1, Y_2\) from \(x_p, x_{3-p}\) to \(y_1, y_2\), respectively. Thus, \(A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A'_0\) and \(B'_1 \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[t_2, b_2]\) show that \(\gamma\) is feasible, a contradiction.

Finally, suppose (iv) fails. Then, since \((G'_0, a_0, b_1, B', b_2)\) is planar and \(y_2\) is the main foot of \(H\), \(G'_0 - (B'[b_1, t_1] \cup B'[b_3, b_4])\) has disjoint paths \(B'_2, A'_0\) from \(b_2, a_0\) to \(y_2, y_1\), respectively. Thus, \(A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A'_0\) and \(B'[b_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'_2\) show that \(\gamma\) is feasible, a contradiction.

\[\square\]

Lemma 4.0.6 \(G'_0\) does not have 3-cuts \(\{a'_0, b'_1, b_2\}\) and \(\{a''_0, b_1, b''_2\}\) with \(b'_1 \in V(B'(b_1, t_1])\) and \(b''_2 \in V(B'[t_2, b_2])\) such that \(\{a'_0, b'_1, b_2\}\) separates \(B'[b'_1, b_2]\) from \(\{a_0, b_1, b_2\}\) and \(\{a'_0, b'_1, b_2\}\) separates \(B'[b'_1, b_2]\) from \(\{a_0, b_1, b_2\}\).

Proof. For, suppose both 3-cuts exist. We choose \(\{a'_0, b'_1, b_2\}\) with \(B'[b_1, b'_1]\) minimal, and choose \(\{a''_0, b_1, b''_2\}\) with \(B'[b''_2, b_2]\) minimal. Then, since \(G'_0\) has a path from \(a_0\) to \(y_1\) and internally disjoint from \(B'\), it follows from Lemma 2.0.8 that

(1) (ii) or (iii) or (iv) of Lemma 2.0.8 holds (and so \(c(A, B') \geq 1\)).

By the minimality of \(B[b_1, b'_1]\) and \(B[b''_2, b_2]\), it follows from (1) and planarity of \((G'_0, a_0, b_1, B', b_2)\) that

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(2) \( G'_0 - B'(b_1, b'_1) - B'(b'_2, b_2) \) has disjoint paths \( B'_1, B'_2, A'_0 \) from \( b_1, b_2, a_0 \) to \( b'_1, b'_2, y_1 \), respectively, which are internally disjoint from \( B' \).

Also by the minimality of \( B[b_1, b'_1] \) and \( B[b'_2, b_2] \), it follows from (iii) and (iv) of Lemma 4.0.5 and Lemmas 2.0.8 and 2.0.9 that

(3) \( G \) has no edge from \( B'(b_1, b'_1) \) to \( A[a_1, x_1] \) or no edge from \( B'(b_1, b'_1) \) to \( A[x_2, a_2] \); and \( G \) has no edge from \( B'(b'_2, b_2) \) to \( A[a_1, x_1] \) or no edge from \( B'(b'_2, b_2) \) to \( A[x_2, a_2] \).

Next, we claim that

(4) \( \alpha(A, B') \leq 1 \).

For, suppose \( \alpha(A, B') = 2 \). Then, by (1), \( a_0 = a'_0 = a''_0 \); so \( c(A, B') \geq 2 \). For convenience, let \( s_1 := b'_1 \) and \( s_2 := b'_2 \). Now, since \( \alpha(A, B') = 2 \), \( G' \) has a path \( A'_i \) (for each \( i \in [2] \)) from \( a_0 \) to \( b_i \) and internally disjoint from \( B' \). Hence, since \( G' \) is 6-connected, \( B'(b_i, s_i) \neq \emptyset \) for \( i \in [2] \).

We claim that there do not exist \( e = ab, e' = a'b' \in E(G) \), such that for some \( i \in [2] \), \( a, a' \in A(a_i, x_i) \), \( b \in B'[b_1, s_1] \), and \( b' \in B'(s_2, b_2) \). For, otherwise, \( \alpha(A, B') = 2 \) and \( c(A, B') = 0 \) by Lemma 3.0.1, because of the path \( B'[b_1, b] \cup e \cup A[a, a'] \cup e' \cup B'[b', b_2] \) from \( b_1 \) to \( b_2 \), the path \( B'_1 \cup B'[b'_1, r_1] \cup R_1 \cup A[x_i, x_{3-i}] \cup P_{i,2} \cup B'[y_2, b'_2] \cup B'_2 \) from \( b_1 \) to \( b_2 \), and the path \( A'_0 \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}] \) from \( a_0 \) to \( a_{3-i} \). This is a contradiction.

Since \( G' \) is 6-connected, \( G \) has at least three pairwise disjoint edges from \( B'(b_i, s_i) \) (for each \( i \in [2] \)) to \( A[a_1, x_1] \cup A[x_2, a_2] \). By (3), for each \( i \in [2] \), we may assume for some \( j \in [2] \), \( G \) has no edge from \( B'(b_i, s_i) \) to \( A[a_j, x_j] \). Now, by symmetry, we assume \( G \) has no edge from \( B'(b_1, s_1) \) to \( A[x_2, a_2] \).

By Lemma 2.0.7, \( G \) has no cross from \( A[a_1, x_1] \) to \( B'(b_1, s_1) \). So, let \( f_i = u_iv_i \) for \( i \in [3] \) be pairwise disjoint edges of \( G \) with \( u_i \in A[a_1, x_1] \) and \( v_i \in B'(b_1, s_1) \), such that \( a_1, u_1, u_3, u_2, a_2 \) occur on \( A \) in order, and \( b_1, v_1, v_3, v_2, b_2 \) occur on \( B' \) in order. We choose \( f_1, f_2 \) so that \( A[u_1, u_2] \cup B'[v_1, v_2] \) is maximal.
Then $G$ has no edge from $B'(s_2, b_2)$ to $A[a_1, x_1]$. For otherwise, $G$ has no edge from $B'(s_2, b_2)$ to $A[x_2, a_2]$ and, hence, has at least three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[a_1, x_1]$. Therefore, $G$ has an edge from $A(a_1, x_1)$ to $B'(s_2, b_2)$, which together with $f_3$ contradicts our claim above.

Thus, $G$ has three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[x_2, a_2]$. Since $G$ has no cross from $A[x_2, a_2]$ to $B'(s_2, b_2)$ (by Lemma 2.0.7), we let $f_j = u_jv_j$ for $j \in \{4, 5, 6\}$ be pairwise disjoint edges of $G$ with $u_j \in A[x_2, a_2]$ and $v_j \in B'(s_2, b_2)$, such that $a_1, u_4, u_6, u_5, a_2$ occur on $A$ in order, and $b_1, v_4, v_6, v_5, b_2$ occur on $B'$ in order. Choose $f_4, f_5$ so that $A[u_4, u_5] \cup B'[v_4, v_5]$ is maximal.

Now by the maximality of $A[u_1, u_2], G$ has an edge $f_7 = u_7v_7$ with $u_7 \in A(u_1, u_2)$ and $v_7 \in B'[t_2, b_2]$, to avoid the cut $\{u_1, u_2, b_1, s_1, a_0\}$ in $G^*$. Similarly, by the maximality of $A[u_4, u_5], G$ has an edge $f_8 = u_8v_8$ with $u_8 \in A(u_4, u_5)$ and $v_8 \in B'[b_1, t_1]$. Now, by the claim above, $v_7 \in B'[t_2, s_2]$ and $v_8 \in B'[s_1, t_1]$. Hence, $f_2, f_4, f_7, f_8$ form a double cross, contradicting Lemma 2.0.7. □

For $i \in [2]$, let $a'_i \in V(A[a_i, x_i])$ with $A[a_i, a'_i]$ minimal such that $a'_i = x_i$ or $G$ has an edge from $a'_i$ to $B'(b'_1, b_2)$. Then $G$ has an edge $e_4 = a_4b_4$ with $a_4 \in A(a'_1, x_1) \cup A[x_2, a'_2]$ and $b_4 \in B[b_1, b'_1]$; for, otherwise, $\{a_0, a'_1, a'_2, b'_1, b_2\}$ would be a 5-cut in $G^*$ separating $H$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. By symmetry, we may assume

\[(5)\quad a_4 \in A(a'_1, x_1).\]

Let $e_3 = a_3b_3 \in E(G)$ with $a_3 = a'_1$ and $b_3 \in B'(b'_1, t_1) \cup B'[t_2, b_2]$. Since $e_3, e_4$ and the paths in $H$ do not form a double cross (by Lemma 2.0.7), we have

\[(6)\quad b_3 \in B'[t_2, b_2].\]

Let $e = ab \in E(G)$ with $a \in A[a_1, a_3]$ and $b \in B'[b_3, b_2]$, such that $B'[b, b_2]$ is minimal, and subject to this, $A[a_1, a]$ is minimal. Further, let $e' = a'b' \in E(G)$ with $a' \in A[a_1, a_4]$ and $b' \in B'[b_1, b_4]$, such that $B'[b_1, b']$ is minimal, and subject to this, $A[a_1, a']$ is minimal.
Similarly, for each $i \in [2]$, let $a''_i \in V(A[a_i, x_i])$ with $A[a_i, a''_i]$ minimal such that $a''_i = x_i$ or $G$ has an edge from $a''_i$ to $B'(b_1, b''_2)$. Since $G^*$ is 6-connected, there exist $j \in [2]$ and $e_6 = a_6b_6 \in E(G)$ such that $a_6 \in A(a'_j, x_j)$ and $b_6 \in B'(b''_2, b_2)$. Since $a''_j \neq x_j$, it follows from Lemma 2.0.7 that there exists $e_5 = a_5b_5 \in E(G)$ such that $a_5 = a''_j$ and $b_5 \in B'(b_1, t_1)$.

(7) $b \in B'(b''_2, b_2)$.

For, otherwise, $b \notin B'(b''_2, b_2)$. Then, $j = 2$ and $a_6 \in A[x_2, a''_2]$ by the choice of $e$. Hence, $b_5 \in B'[b_1, b_4]$ to avoid the double cross $e_3, e_4, e_5, e_6$. So $b_5 = b_1$ by (3), and thus $a_5 \neq a_2$.

Let $e_2^* = a_2b_2^* \in E(G)$. Then $b_2^* \in B'[b_6, b_2]$ to avoid the double cross $e_5, e_2^*, e_3, e_6$.

Note that $a_5 \neq x_2$. Then $\alpha(A, B') = 2$ by Lemma 3.0.1 and the following paths: the path $A[a_5, a_2] \cup e_5$ from $a_2$ to $b_1$, the path $e_2^* \cup B'[b_2^*, b_2]$ from $a_2$ to $b_2$, the path $B'_1 \cup B'[b'_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup P_{2,2} \cup B'[y_2, b''_2] \cup B^*_2$ from $b_1$ to $b_2$, and the path $A_0^* \cup P_{1,1} \cup A[a_1, x_1]$ from $a_0$ to $a_1$. This is a contradiction to (4). \(\square\)

If $a' \neq x_1$ then $\alpha(A, B') = 2$ by Lemma 3.0.1 and the following paths: the path $A[a_1, a'] \cup e' \cup B'[b_1, b']$ from $a_1$ to $b_1$, the path $A[a_1, a] \cup e \cup B'[b_2, b_2]$ from $a_1$ to $b_2$, the path $B^*_1 \cup B'[b'_1, r_1] \cup R_1 \cup A[x_1, x_2] \cup P_{1,2} \cup B'[y_2, b''_2] \cup B^*_2$ from $b_1$ to $b_2$, and the path $A_0^* \cup P_{2,1} \cup A[x_2, a_2]$ from $a_0$ to $a_2$. This contradicts (4).

So $a' = x_1$. Hence, by the choice of $e'$ and Lemma 2.0.7, $G$ has no edge from $A[a_1, x_1]$ to $B'[b_1, t_1]$. Thus, $G$ has an edge from $a_1$ to $B'[t_2, b_2]$. So by the choice of $e$ and by Lemma 2.0.7, $a = a_1$ and, hence, $b \neq b_2$.

We claim $a_6 \in A[x_2, a''_2]$. For, otherwise, $a_6 \in A(a''_1, x_1]$. Then $a_5 \in A[a_1, x_1]$. Now, $e_5$ contradicts the choice of $e'$, or $e_5, e', P_{1,2}, P_{2,1}$ form a double cross, contradicting Lemma 2.0.7.

Thus, by (3), $b_5 = b_2$. So $b_5 \in B'[b_1, b']$ to avoid the double cross $e, e', e_5, e_6$.

Suppose $H$ contains disjoint paths $Y_1, Y_2$ from $x_1, x'_2$ to $y_1, y_2$, respectively, and internally disjoint from $A$. Then $\alpha(A, B') = 2$ by Lemma 3.0.1 and the following paths: the
path $A[a_5, a_2] \cup e_5 \cup B'[b_1, b_5]$ from $a_2$ to $b_1$, the path $A[a_6, a_2] \cup e_6 \cup B'[b_6, b_2]$ from $a_2$ to $b_2$, the path $B^*_1 \cup B'[b_1', r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[y_2, b_2'] \cup B_2^*$ from $b_1$ to $b_2$, and the path $A_0^* \cup Y_1 \cup A[a_1, x_1]$ from $a_0$ to $a_1$. This contradicts (4).

So by Lemma 4.0.2, $H$ has disjoint paths $Y_1, Y_2$ from $x_2, x_1'$ to $y_1, y_2$, respectively, and internally disjoint from $A$. We have a contradiction to (4) as $\alpha(A, B') = 2$ because of Lemma 3.0.1 and the following paths: the path $A[a_1, x_1] \cup c' \cup B'[b_1, b']$ from $a_1$ to $b_1$, the path $A[a_1, a] \cup e \cup B'[b, b_2]$ from $a_1$ to $b_2$, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[y_2, b_2'] \cup B_2^*$ from $b_1$ to $b_2$, and the path $A_0^* \cup Y_1 \cup A[x_2, a_2]$ from $a_0$ to $a_2$.  

\[\square\]

**Lemma 4.0.7** Let $\{a_0', b_1', b_2'\}$ be a cut in $G_0'$ separating $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$, with $b_1' \in B'[b_1', t_1]$ and $b_2' \in B[x_2, b_2]$. Then $b_1' = b_1$, $b_2' \neq b_2$, $a_0' = a_0$, $y_1$ is a cut vertex in $G_0'$ separating $b_2$ from $\{a_0, b_1\}$, $b_2$ has degree 1 in $G_0'$, and for some $p \in [2]$, $G$ has an edge from $b_2$ to $x_p$ and no edge from $b_2$ to $A - x_p$.

**Proof.** For $i \in [2]$, let $a_i' \in V(A[a_i, x_i])$ with $A[a_i, a_i']$ minimal such that $a_i' = x_i$ or $G$ has an edge from $a_i'$ to $B'(b_1', b_2')$. Since $G^*$ is 6-connected, there exist $i, j \in [2]$ such that $G$ has an edge $e_4 = a_4b_4$ with $a_4 \in A(a_i', x_i)$ and $b_4 \in B'[b_j, b_j']$. By symmetry, assume $i = 1$. Then $a_1' \neq x_1$ and let $e_3 = a_3b_3 \in E(G)$ such that $a_3 = a_1'$ and $b_3 \in B'(b_1', t_1) \cup B'[t_2, b_2')$. Now $b_3 \in B'[t_3-j, b_3-j]$, to avoid the double cross formed by $e_3, e_4$ and two paths in $H$ (by Lemma 2.0.7).

First, we show that

(1) $b_1' = b_1$.

For, suppose $b_1' \neq b_1$. Choose the 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \neq b_1$, such that $B[b_2, b_2]$ is minimal and, subject to this, $B[b_1, b_1']$ is minimal.

Observe that $b_4 \in B[b_1, b_1']$. For, otherwise, $b_4 \in B(b_2, b_2)$. Then $b_3 \in B(b_1', t_1]$. Now, by Lemma 2.0.9 and (ii) of Lemma 4.0.5, $G_0'$ has a 3-cut contradicting the choice of $\{a_0', b_1', b_2'\}$.
Then \( b_3 \in B'[t_2, b'_2] \). Hence, because of \( e_3, e_4 \), it follows from (i) of Lemma 4.0.5 that \( G'_0 \) has a 3-cut \( \{a''_0, b'_1, b''_2\} \) with \( b''_1 \in B'[b_1, b_4] \) and \( b''_2 \in B'[t_2, b_5] \), separating \( B'[b'_1, b'_2] \) from \( \{a_0, b_1, b_2\} \). By Lemma 2.0.8 and the choice of \( \{a'_0, b'_1, b'_2\} \), we have \( b''_1 = b_1 \).

By Lemma 4.0.6, \( b'_2 \neq b_2 \). Hence, by Lemma 2.0.8, there exists \( a''_0 \in V(G'_0) \), such that \( \{b''_1, b'_2, a''_0\} \) is a 3-cut in \( G'_0 \) separating \( \{a_0, b_1, b_2\} \) from \( B'[b'_1, b'_2] \). For \( i \in [2] \), let \( a''_i \in A[a_i, x_i] \) with \( A[a_i, a''_i] \) minimal such that \( a''_i = x_i \) or \( G \) has an edge from \( a''_i \) to \( B'(b'_1, b'_2) \).

Since \( G^* \) is 6-connected, there exist \( k \in [2] \) and \( e_5 = a_5b_5 \in E(G) \) with \( a_5 \in A(a''_k, x_k) \) and \( b_5 \in B'(b'_2, b_2) \). Let \( e_6 = a_6b_6 \in E(G) \) with \( a_6 = a''_k \) and \( b_6 \in B'(b'_1, t_1) \cup B'[t_2, b'_2] \). Then \( b_6 \in B'(b'_1, t_1) \), to avoid the double cross formed by \( e_5, e_6 \) and two paths in \( H \). Because of \( e_5 \) and \( e_6 \), it follows from (ii) of Lemma 4.0.5 and the choice of \( \{a'_0, b'_1, b'_2\} \) that \( G'_0 \) has a 2-cut \( \{y_1, b'_2\} \) with \( b'_2 \in B'[b_5, b_2] \), separating \( B'[y_1, b''_2] \) from \( \{a_0, b_1, b_2\} \). But then, by Lemma 2.0.9, \( \{y_1, b'_2\} \) and \( \{a'_0, b'_1, b'_2\} \) force a 3-cut in \( G'_0 \), which contradicts the choice of \( \{a'_0, b'_1, b'_2\} \).  

Since \( G^* \) is 6-connected, it follows from (1) that \( b_2 \neq b'_2 \). We choose \( \{a'_0, b'_1, b'_2\} \) so that \( B[b_2, b'_2] \) is minimal. Then, by (1) and (ii) of Lemma 4.0.5, \( G'_0 \) has a 2-cut \( \{y_1, b''_2\} \) with \( b''_2 \in B'[b_4, b_2] \), separating \( B'[y_1, b''_2] \) from \( \{a_0, b_1, b_2\} \).

Moreover, \( b''_2 = b_2 \); for, otherwise, by Lemma 2.0.9, \( \{y_1, b''_2\} \) and \( \{a'_0, b'_1, b'_2\} \) force a 3-cut in \( G'_0 \), which contradicts the choice of \( \{a'_0, b'_1, b'_2\} \). Hence, \( y_1 \) is a cut vertex in \( G'_0 \) separating \( b_2 \) from \( \{a_0, b_1\} \) and \( \alpha(A, B') \leq 1 \). And (for any choice of \( \{a'_0, b'_1, b'_2\} \)), \( a'_0 = a_0 \); or else, since \( y_1 \) is a cut vertex in \( G'_0 \) separating \( b_2 \) from \( \{a_0, b_1\}, \{b_1, a'_0, b'_2, b_2\} \) is a cut in \( G \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction.

So by (1), \( G'_0 - V(B'(b_1, t_1) \cup B'(y_1, b_2)) \) has disjoint paths \( B^*_1, A^*_0 \) from \( b_1, a_0 \) to \( t_1, y_1 \), respectively, such that \( A^*_0 \) is internally disjoint from \( B' \). By the choice of \( \{a'_0, b'_1, b'_2\} \), \( G'_0 - V(B'(b'_2, b_2)) \) has a path \( B^*_2 \) from \( b_2 \) to \( b'_2 \).

(2) For \( i \in [2] \), if \( G \) has an edge from \( B'(b'_2, b_2) \) to \( A[a_i, x_i] \), then \( G \) has no edge from \( A[a_i, x_i] \) to \( B'[b_1, t_1] \).
For, suppose for some \( i \in [2] \), \( G \) has an edge \( e \) from \( b \in B'[b_2', b_2] \) to \( a \in A[a_i, x_i] \) and an edge \( e' \) from \( a' \in A[a_i, x_i] \) to \( b' \in B'[b_1, t_1] \).

Then, \( \alpha(A, B') = 2 \) by Lemma 3.0.1 and the following paths: \( A[a_i, a'] \cup e' \cup B'[b_1, b'] \) from \( a_i \) to \( b_1 \), the path \( A[a_i, a] \cup e \cup B'[b, b_2] \) from \( a_i \) to \( b_2 \), the path \( B^*_1 \cup R_1 \cup A[x_i, x_{3-i}] \cup P_{i,2} \cup B^*_2 \) from \( b_1 \) to \( b_2 \), and the path \( A^*_0 \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}] \) from \( a_0 \) to \( a_{3-i} \). This is a contradiction. \( \square \)

(3) \( B'(b_2', b_2) = \emptyset \), and so \( b_2 \) has degree 1 in \( G_0' \).

For, suppose \( B'(b_2', b_2) \neq \emptyset \). Then, as \( G^* \) is 6-connected, \( G \) has edges from \( B'(b_2', b_2) \) to \( A[a_1, x_1] \cup A[x_2, a_2] \).

Indeed, \( G \) has an edge \( e_3 \) from \( B'(b_2', b_2) \) to \( A[a_1, x_1] \), and an edge \( e_4 \) from \( B'(b_2', b_2) \) to \( A[x_2, a_2] \). For otherwise, there exists \( i \in [2] \), such that all edges of \( G \) from \( B'(b_2', b_2) \) to \( A \) end in \( A[a_i, x_i] \). Let \( u_1, u_2 \in V(A[a_i, x_i]) \), such that \( G \) has edges from \( B'(b_2', b_2) \) to \( u_1, u_2 \), respectively, and, subject to this, \( A[u_1, u_2] \) is maximal. Now, by Lemma 2.0.7, \( G \) has no edge from \( A(u_1, u_2) \) to \( B'[t_2, b_2] \). Moreover, by (2), \( G \) has no edge from \( A(u_1, u_2) \) to \( B'[t_1, b_1] \). But then, \( \{t_1, u_1, u_2, b_2, b_2\} \) is a cut in \( G \) separating \( V(A[u_1, u_2] \cup B'[b_2, b_2]) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Now \( A[a_1, x_1] \cup e_3 \cup B'(b_2', b_2) \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A^*_0 \cup B'[b_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[y_2, b_2'] \cup B^*_2 \) show that \( \gamma \) is feasible, a contradiction. \( \square \)

(4) \( G \) has no edge from \( b_2 \) to \( A[a_1, x_1] \cup A(x_2, a_2) \).

Suppose for some \( i \in [2] \), \( G \) has an edge \( e \) from \( b_2 \) to \( a \in A[a_i, x_i] \). Let \( e' = a_1b' \in E(G) \) with \( b' \neq t_1 \). Obviously, \( b' \notin B'[t_2, b_2] \); otherwise, \( e, e' \) and two disjoint paths in \( H \) force a double cross, contradicting Lemma 2.0.7.

So \( b' \in B[b_1, t_1] \). Now \( \alpha(A, B') = 2 \) by Lemma 3.0.1 and the following paths: the path \( e' \cup B'[b_1, b'] \) from \( a_i \) to \( b_1 \), the path \( A[a_i, a] \cup e \) from \( a_i \) to \( b_2 \), the path \( B^*_1 \cup R_1 \cup A[x_i, x_{3-i}] \cup P_{i,2} \cup B'[y_2, b_2] \) from \( b_1 \) to \( b_2 \), and the path \( A^*_0 \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}] \) from \( a_0 \) to \( a_{3-i} \). However, this is a contradiction. \( \square \)
Now, since the degree of $b_2$ in $G$ is at least 2, it follows from (4) that $G$ has an edge from $b_2$ to $x_p$ for some $p \in [2]$. If $G$ has no edge from $b_2$ to $x_{3-p}$ then we are done. So assume $b_2x_1, b_2x_2 \in E(G)$. Then $a_1 \neq x_1$ and $a_2 \neq x_2$. Now, by Lemma 2.0.7, $G$ has no edge from $\{a_1, a_2\}$ to $B'[t_2, b_2)$. Since $G^*$ is 6-connected, $G$ has edges $e_1, e_2$ from $B'(b_1, t_1)$ to $a_1, a_2$, respectively. But then, it follows from (iii) of Lemma 4.0.5 that $G'_0$ contains a 3-cut, which contradicts (1).

Lemma 4.0.8 $H$ is the unique main $A-B'$ core in $\gamma$.

Proof. Suppose for a contradiction that $H''$ is a main $A-B'$ core with $H'' \neq H$, and let $w_1, w_2$ be the feet of $H''$ (with $w_2$ as the main foot). Then, by Lemma 2.0.7, $w_2 = r_1$ and $b_1, w_2, w_1, y_1, y_2, b_2$ occur on $B'$ in order.

Recall that the definition of $x'_i, X'_i$ before Lemma 4.0.2. For $i \in [2]$, let $x''_i \in V(A(x_1, x_2))$ such that $x''_i, x_i$ are incident with a common finite face of $H'' - w_1$, and $H'' - w_1$ has a path from $x''_i$ to $w_2$ and internally disjoint from $A$. So for $i \in [2]$, let $X''_i$ be the path from $w_2$ to $x''_i$ on the outer walk of $H'' - \{w_1, x_i\}$ without going through $x_{3-i}$, and, moreover, let $X^*_i$ be the path from $x_i$ to $w_2$ on the outer walk of $H'' - w_1$ without going through $x_{3-i}$.

Suppose $H$ contains disjoint paths from $y_1, y_2$ to $x_2, x'_1$, respectively, and internally disjoint from $A$, as well as disjoint paths from $y_1, y_2$ to $x_1, x'_2$, respectively, and internally disjoint from $A$. Then, by Lemma 2.0.7, for any $i \in [2]$, $H''$ does not contain disjoint paths from $w_1, w_2$ to $x_i, x''_{3-i}$, respectively, and internally disjoint from $A$. This contradicts (iii) of Lemma 4.0.2.

Hence, by symmetry, we may assume that $H$ contains no disjoint paths from $y_1, y_2$ to $x_1, x'_2$, respectively, and internally disjoint from $A$. Then by Lemma 4.0.2, $H$ contains disjoint paths $Y'_1, Y'_2$ from $y_1, y_2$ to $x_2, x'_1$, respectively, and internally disjoint from $A$.

Then by Lemma 2.0.7 and 4.0.2, we may further assume $H''$ contains disjoint paths $Y''_1, Y''_2$ from $w_1, w_2$ to $x_2, x'_1$, respectively, and internally disjoint from $A$, but no disjoint paths from $w_1, w_2$ to $x_1, x''_2$, respectively, and internally disjoint from $A$. Moreover, by
(i) of Lemma 4.0.2, \( H - \{y_1, y_2\} \cup V(A(x_1, x_2)) \) contains a path \( D' \) from \( x_1 \) to \( x_2 \), and
\( H'' - \{w_1, w_2\} \cup V(A(x_1, x_2)) \) contains a path \( D'' \) from \( x_1 \) to \( x_2 \).

(1) There is no \( A-B' \) path in \( G \) from \( A(x_1, x_2) \) to \( B'(w_1, y_1) \).

For, suppose that \( P \) is an \( A-B' \) path from \( p \in V(A(x_1, x_2)) \) to \( p' \in V(B'(w_1, y_1)) \). Then \( G'_0 - B'(w_2, w_1) - B'[y_2, b_2] \) does not contain disjoint paths \( B'_1, A'_0 \) from \( b_1, a_0 \) to \( p', y_1 \), respectively; otherwise, \( A[a_1, x_1] \cup D'' \cup A[x_2, a_2] \cup Y'_1 \cup A'_0 \) and \( B'_1 \cup P \cup A(x_1, x_2) \cup Y'_2 \cup B'[y_2, b_2] \) show that \( \gamma \) is feasible, a contradiction. Hence, there exists \( w' \in V(B'(w_2, w_1)) \), \( a'_0 \in V(G'_0) \), and \( b'_2 \in V(B'[y_2, b_2]) \), such that \( \{w', a'_0, b'_2\} \) is a 3-cut in \( G'_0 \) separating \( B'[w', b'_2] \) from \( \{a_0, b_1, b_2\} \).

Now \( b_1 = w_2 \). For, suppose not. Since \( w_1, w_2 \) are feet of \( H'' \), \( w_1, w_2 \) are incident with a common finite face of \( G'_0 \). Therefore, \( \{w_2, a'_0, b'_2\} \) is a 3-cut in \( G'_0 \) separating \( B'[w_2, b'_2] \) from \( \{a_0, b_1, b_2\} \), a contradiction to Lemma 4.0.7. Similarly, by the symmetry between \( H \) and \( H'' \), we can also prove \( b_2 = y_2 \).

Now, since \( b'_2 \in V(B'[y_2, b_2]) \), \( b'_2 = b_2 \). So \( a'_0 = a_0 \); or else, \( \{b_1, a'_0, b'_2\} \) is a 3-cut in \( G'_0 \) separating \( a_0 \) from \( B'(b_1, b_2) \), a contradiction. Then \( a_0, b_1, w', w_1 \) are incident with a common finite face of \( G'_0 \). Similarly, by the symmetry between \( H \) and \( H'' \), \( a_0, b_2, y_1 \) are incident with a common finite face of \( G'_0 \), which implies \( \alpha(A, B') = 0 \).

By Lemma 4.0.2, \( V(X'_2 \cap X'_1) - \{w_2\} = \emptyset \). Now \( \alpha(A, B') \geq 1 \) by Lemma 3.0.1 and the following paths: the path \( A'_0 \cup Y'_1 \cup A[x_2, a_2] \) from \( a_0 \) to \( a_2 \), the path \( X'_2 \cup A(x_1, x_2) \cup Y'_2 \) from \( b_1 \) to \( b_2 \), and the path \( A[a_1, x_1] \cup X'_1 \) from \( a_1 \) to \( b_1 \). This is a contradiction. \( \square \)

(2) \( a_1 = x_1 \) and \( a_2 = x_2 \).

Recall that for \( i \in [2] \), \( P_{1,i} \) and \( P_{2,3-i} \) are disjoint paths from \( x_1, x_2 \) to \( y_i, y_{3-i} \), respectively, in \( H - A(x_1, x_2) \). For \( i \in [2] \), let \( Q_{1,i}, Q_{2,3-i} \) be disjoint paths from \( x_1, x_2 \) to \( w_i, w_{3-i} \), respectively, in \( H'' - A(x_1, x_2) \).

We claim that for \( i \in [2] \), \( G \) has no edge from \( A[a_i, x_i] \) to \( B'(b_i, w_2) \). For, suppose there exists \( e' = a'b' \in E(G) \) with \( a' \in A[a_i, x_i] \) and \( b' \in B'(b_i, w_2) \). Then \( b_i \neq w_2 \).
By Lemma 4.0.7, \( G'_0 - B'[b', w_2] - B'[y_2, b_2] \) contains disjoint paths \( B^*_1, A^*_0 \) from \( b_1, a_0 \) to \( w_1, y_1 \), respectively. Now \( A[a_i, x_i] \cup e' \cup B'[b', w_2] \cup Q_{3-i,2} \cup A[x_{3-i}, a_{3-i}] \cup P_{3-i,1} \cup A^*_0 \) and \( B^*_1 \cup Q_{i,1} \cup P_{i,2} \cup B'[y_2, b_2] \) show that \( \gamma \) is feasible, a contradiction.

Due to the symmetry between \( H \) and \( H'' \), with the same argument above, we can show that for \( i \in [2] \), \( G \) has no edge from \( A[a_i, x_i] \) to \( B'[y_2, b_2] \). Thus, (2) follows from Lemma 4.0.4 and the assumption that \( G^* \) is 6-connected. \( \square \)

(3) \( H'' - V(X_1^* \cup X_2^*) \) contains a path \( Q'' \) from \( w_1 \) to \( A(x_1, x_2) \); and \( H - V(X_1 \cup X_2) \) contains a path \( Q \) from \( y_1 \) to \( A(x_1, x_2) \).

By the symmetry between \( H \) and \( H'' \), we only prove the existence of \( Q'' \). Suppose for a contradiction that \( Q'' \) does not exist.

We see that \( (N(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2)) \). For, otherwise, by (ii) of Lemma 4.0.2, there exists \( v'' \in N(w_1) \cap V(H'') \), \( c_1'' \in A(x_1, x_2'') \), and \( c_2'' \in X_2''(x_2'', w_2) \), such that \( v'' \notin X_2'' \cup A(x_1, x_2) \), \{\( c_1'', c_2'' \)\} is a cut in \( H'' - \{w_1, x_2\} \) separating \( v'' \) from \( x_1 \), and there exists a path \( P_1'' \) from \( v'' \) to \( c_1'' \) in \( H'' \), which is internally disjoint from \( X_2'' \cup A[x_1, x_2'] \). But then, \( w_1 v'' \cap P_1'' \) is a path from \( w_1 \) to \( A(x_1, x_2) \) in \( H'' - V(X_1^* \cup X_2^*) \), a contradiction.

Now, since \( Q'' \) does not exist, combined with \( (N(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2)) \), we may further assume \( (N(w_1) \cap V(H'')) \subseteq V(X_2^*) \), contradicting (iii) of Lemma 4.0.1. \( \square \)

(4) \( b_1 = w_2 \) and \( b_2 = y_2 \).

By the symmetry between \( H \) and \( H'' \), we only show \( b_1 = w_2 \). Suppose for a contradiction that \( b_1 \neq w_2 \).

Since \( w_1, w_2 \) are incident with a common finite face of \( G'_0 \), it follows from Lemma 4.0.7 that \( G'_0 - B'[w_2, w_1] - B'[y_2, b_2] \) contains disjoint paths \( B^*_1, A^*_0 \) from \( b_1, a_0 \) to \( w_1, y_1 \), respectively.
Now, \(A[a_1, x_1] \cup X_1^* \cup X_2^* \cup A[x_2, a_2] \cup Y_1' \cup A_0'\) and \(B_1^* \cup Q'' \cup A(x_1, x_2) \cup Y_2' \cup B'[y_2, b_2]\) show that \(\gamma\) is feasible, a contradiction. \(\square\)

Note that \(G\) has no \(A-B'\) path from \(a_1\) to \(B'(w_1, y_1)\), as such a path together with \(Y_2'', Y_1', Y_2'\) forms a double cross, contradicting Lemma 2.0.7. So by (1) and (4), \(\{b_1, b_2, w_1, y_1, a_2\}\) is a cut in \(G\) separating \(a_0\) from \(a_1\), a contradiction. \(\square\)

We now use \(A, B'\) to form a new frame \(A', B'\), called core frame.

**Lemma 4.0.9** Let \(M_0\) denote the union of all the \(A-B'\) bridges that are disjoint from \(H - A - y_1\). Then there exists an induced path \(A' \subseteq (A \cup M_0) - B'\) from \(a_1\) to \(a_2\) in \(G\), such that \(A'[a_i, x_i] = A[a_i, x_i]\) for \(i \in [2]\) and the following hold:

(i) \(A', B'\) is a good frame in \(\gamma\).

(ii) Each \(A'-B'\) bridge lying on \(B'[r_1, y_1]\) is contained in some \(A-B'\) bridge.

(iii) There exists an induced subgraph \(H^*\) in \(G\), such that \(A'[x_1, x_2] \cup H \subseteq H^*\), all \(A'-B'\) bridges not lying on \(B'[r_1, y_1]\) are contained in \(H^*\), and \(H^*\) is separated from \(\{a_0, a_1, a_2, b_1, b_2\}\) by \(V(A'[x_1, x_2]) \cup \{y_1, y_2\}\) in \(G\).

(iv) For any \(v \in (V(H^*) - V(A') \cup \{y_1\})\), \(H^* - y_1\) contains a path from \(v\) to \(y_2\) and internally disjoint from \(A'\).

(v) If \(l, r\) are the extreme hands of an \(A'-B'\) bridge lying on \(B'[r_1, y_1]\) then \(\{l, r\} \neq \{x_1, x_2\}\), and \(H^* - y_1\) does not contain a path from \(y_2\) to \(A'(l, r)\) and internally disjoint from \(A'\).

**Proof.** We choose the induced path \(A'\) so that \(A' \subseteq A \cup M_0 - B'\) is from \(a_1\) to \(a_2\), such that \(A'[a_i, x_i] = A[a_i, x_i]\) for \(i \in [2]\), (i)-(iv) are satisfied, and, subject to this, \(H\) is maximal. Note that such \(A'\) exists, as \(A\) satisfies (i)-(iv).

To prove (v), let \(M\) be an \(A'-B'\) bridge \(M\) lying on \(B'[r_1, y_1]\) with extreme hands \(l, r\) and feet \(l', r'\). If \(\{l, r\} = \{x_1, x_2\}\) then, since \(M\) is contained in an \(A-B'\) bridge (by (ii)), \(M\)
is contained in a main $A$-$B'$ core, a contradiction to Lemma 4.0.8. Hence, $H - y_1$ contains a path $Y_2$ from $y_2$ to $y'_2 \in A'(l, r)$ and internally disjoint from $A'$.

Let $T$ be an induced path in $M - V(A'(l, r) \cup B'[l', r'])$ from $l$ to $r$, and let $C_1, C_2, \ldots, C_n$ be the components of $M \cup A'[l, r] \cup B'[l', r'] - V(T)$ not containing $A'(l, r)$ and not containing $B'[l', r']$. We choose $T$, such that $|T| := (|V(\bigcup_{i \in [n]} C_i)|, |V(C_1)|, |V(C_2)|, \ldots, |V(C_n)|)$ is maximal with respect to the lexicographical ordering.

We claim $n = 0$. For, suppose $n > 0$. Let $l_n, r_n \in N(C_n) \cap V(T)$ such that $T[l_n, r_n]$ is maximal. Since $G^*$ is 6-connected, there exists another component $C$ of $(M \cup A'[l, r] \cup B'[l', r']) - V(T)$, such that $N(C) \cap T(l_n, r_n) \neq \emptyset$. Now, let $T'$ be an induced path in $G[T \cup C_n]$ between $l_n$ and $r_n$, such that $T' \cap T(l_n, r_n) = \emptyset$. Clearly, $|T'| > |T|$, a contradiction.

Now, let $A''$ be obtained from $A'$ by replacing $A'[l, r]$ with $T$. Clearly, $A''[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$. Since $T$ is induced, $A''$ is induced. Moreover, since $n = 0$, then any component of $G[V(M \cup A'[l, r] \cup B'[l', r'])] - T$ contains $A'(l, r)$ or $B'[l', r']$, and so $G - V(A'')$ is connected. Hence, $A''$, $B'$ is a frame. Since $A''_0(B') = A'_0(B') = A_0(B')$, we see that $A''$, $B'$ is a good frame in $\gamma$.

Next, we show that $G$ has no $A'$-$B'$ path from $A'(l, r)$ to $B'[b_1, y_1]$ and disjoint from $T$. For otherwise, let $S$ be an $A'$-$B'$ path from $s \in A'(l, r)$ to $s' \in B'[b_1, y_1]$ and disjoint from $T$. Then $A''$ and $B'[b_1, s'] \cup S \cup A'[s, y_2] \cup Y_2 \cup B'[y_2, b_2]$ are disjoint paths from $a_1, b_1$ to $a_2, b_2$, respectively, in $G - V(A_0(B') - B') - y_1$, a contradiction to (i) of Lemma 3.0.2.

Hence, there does not exist an $A'$-$B'$ bridge $N$ lying on $B'[r_1, y_1]$, such that $N \neq M$, $N \cap A'(l, r) \neq \emptyset$, and $N \cap B'[b_1, y_1] \neq \emptyset$. So each $A''$-$B'$ bridge lying on $B'[r_1, y_1]$ must be contained in some $A'$-$B'$ bridge and, hence, contained in some $A$-$B'$ bridge. So $A''$, $B'$ satisfies (ii).

Moreover, $V(A''[x_1, x_2]) \cup \{y_1, y_2\}$ is a cut in $G$ separating $V(H)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Now, we let $V''$ be the set of vertices of $A'' \cup B'[b_1, y_1] \cup B'[y_2, b_2]$-bridge of $G$ containing $A'(l, r)$, and let $H'' := G[V'' \cup V(A''[x_1, x_2])]$. Then clearly (iii) and (iv) holds for $A''$, $B'$.

However, $H''$ properly contains $H$, a contradiction. □
CHAPTER 5
INSIDE THE MAIN $A'-B'$ CORE

We use the notation of the previous chapter: $\gamma$ is infeasible, $A', B'$ is a core frame, and let $H' := H^* - \{x_1y_2, x_2y_2\}$ with extreme hands $x_1, x_2$ and feet $y_1, y_2$ (such that $y_2$ is the main foot), where $B'$ is defined as in Lemma 4.0.3, $A', H^*, x_1, x_2, y_1, y_2$ are defined as in Lemma 4.0.9, and $t_1, t_2, R_1, r_1$ are defined after Lemma 4.0.3. And we say that $H'$ is the main $A'-B'$ core in $\gamma$.

We now study the structure of $G$ inside the main $A'-B'$ core $H'$.

**Lemma 5.0.1** $(H' - y_1, A'[x_1, x_2], y_2)$ is planar, the degree of $y_2$ in $H' - y_1$ is at least 2, and $H' - y_1 - A'(x_1, x_2)$ contains disjoint paths from $y_1, y_2$ to $x_i, x_{3-i}$, respectively, for $i \in [2]$. Moreover, for $i \in [2]$, let $X_i$ be the path from $x_i$ to $y_2$ on the outer walk of $H' - y_1$ without going through $x_3 - i$, then $N(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i)$ for $i \in [2]$.

**Proof.** We can apply the same proof in Lemma 3.0.4, and show that $(H' - y_1, A'[x_1, x_2], y_2)$ is planar, and $N(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i)$ for $i = 1, 2$.

Moreover, since $V(H - y_1) \subseteq V(H' - y_1)$, then, by (iii) of Lemma 4.0.1, the degree of $y_2$ in $H' - y_1$ is at least 2, and $H' - A'(x_1, x_2) - \{y_1x_1, y_1x_2\}$ contains disjoint paths from $y_1, y_2$ to $x_1, x_2$, respectively, as well as disjoint paths from $y_1, y_2$ to $x_2, x_1$, respectively. □

**Lemma 5.0.2** Let $R$ be an $A'-B'$ path from $r \in A'(x_1, x_2)$ to $r' \in B'[r_1, y_1)$ such that $B'[r_1, r']$ is minimal. If $r' \neq r_1$ then the following conclusions hold:

(i) There exists an $A-B$ core $H_1$ with $r_1$ as a foot.

(ii) Let $r_2$ be the other foot of $H_1$, then there exists an $A'-B'$ bridge with $r_1$ as a foot, intersecting $A'$ only at $x_j$ for some $j \in [2]$, and lying on $B'[r_1, r_2]$. 

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Lemma 4.0.9). Hence, since \( r_j \in r \) which must have \( r \) by Lemma 4.0.8), then by the definition of \( r \) Lemma 4.0.1, \( G \) has an \( A'-B' \) bridge with feet \( l'_1, r'_1 \), which is internally disjoint from \( R \) and intersecting \( A' \) only at \( x_j \), such that \( r' \in B'(l'_1, r'_1) \).

(iv) If \( G_0' \) has a cut \( \{a'_0, b'_1, b'_2\} \) separating \( B'[b'_1, b'_2] \) from \( \{a_0, b_1, b_2\} \) such that \( b'_1 \in B'(r_1, r') \) and \( b'_2 \in B'[y_2, b_2] \), then \( r_1 = b_1 \) and \( a'_0 = a_0 \); \( G_0' \) has no path from \( a_0 \) to \( b_1 \) and internally disjoint from \( B' \), and \( \alpha(A', B') \leq 1 \).

**Proof.** To prove (i), assume that \( r_1 \) is not a foot of any \( A-B \) core. Then by the definition of \( r_1 \), \( G \) has an edge from \( r_1 \) to \( a' \in V(A(x_1, x_2)) \). Since \( r' \neq r_1 \), \( a' \notin A'(x_1, x_2) \). Moreover, \( a' \) is not contained in any \( A'-B' \) bridge lying on \( B'[r_1, y_1] \), as any such \( A'-B' \) bridge is contained in an \( A-B' \) bridge (by (ii) of Lemma 4.0.9). So \( a' \in V(H' - y_1) \setminus V(A') \). Hence, by (iv) of Lemma 4.0.9, \( H' - y_1 \) has a path \( Y_2 \) from \( a' \) to \( y_2 \) and internally disjoint from \( A' \). Therefore, \( A' \) and \( B'[b_1, r_1] \cup r_1a' \cup Y_2 \cup B'[y_2, b_2] \) are disjoint paths from \( a_1, b_1 \) to \( a_2, b_2 \), respectively, in \( G - V(A'_0(B') - B') \cup \{y_1\} \), contradicting (i) of Lemma 3.0.2.

Now, we prove (ii). By Lemma 4.0.4, \( r_2 \) is the main foot of \( H_1 \). Hence, by (iii) of Lemma 4.0.1, \( r_1 \) has two neighbors \( u_1, u_2 \) in \( H_1 - r_2 - A \). Since \( B'[r_1, r_2] \) is induced in \( G - \{r_1r_2\} \) by Lemma 4.0.3, \( u_p \notin B' \) for some \( p \in [2] \). Moreover, \( u_p \notin A'(x_1, x_2) \) since \( r' \neq r_1 \). Thus, \( u_p \) must be contained in some \( A'-B' \) bridge \( M_0 \) lying on \( B'[r_1, r_2] \), which must have \( r_1 \) as a foot and cannot have both \( x_1 \) and \( x_2 \) as extreme hands (by (v) of Lemma 4.0.9). Hence, since \( r' \neq r_1 \), this \( A'-B' \) bridge intersect \( A' \) only at \( x_j \) for some \( j \in [2] \).

Obviously, since \( G^* \) is 6-connected, \( r' \in B'(r_1, r_2) \) to avoid the cut \( \{r_1, r_2, x_1, x_2\} \) in \( G^* \) separating \( V(H_1) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \). Let \( l'_0, r'_0 \) be the feet of \( M_0 \) with \( l'_0 = r_1 \) and \( r'_0 \in B'[r_1, r_2] \). For, suppose (iii) fails. Then \( r' \in B'[r'_0, r_2] \). Since \( x_{3-j} \notin V(H_1 \cap A') \) by Lemma 4.0.8), then by the definition of \( r' \), \( \{x_j, r_1, r'\} \) is a cut in \( G \) separating \( M_0 \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

To prove (iv), we observe that \( B'[r_1, r_2] \) is on the boundary of a finite face of \( G_0' \). Therefore, since \( r' \in B'(r_1, r_2) \), \( a'_0 \) and \( r_1 \) are also incident with that finite face. Suppose \( r_1 \neq b_1 \) or \( a'_0 \neq a_0 \). Then \( \{a'_0, r_1, b'_2\} \) is a 3-cut in \( G_0' \) separating \( B'[r_1, b'_2] \) from \( \{a_0, b_1, b_2\} \).
By Lemma 4.0.7, \( r_1 = b_1 \). So \( a'_0 \neq a_0 \). Then, by Lemma 4.0.7, \( \{a'_0, b_1, b'_2, b_2\} \) is a cut in \( G \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction. So, \( r_1 = b_1 \) and \( a'_0 = a_0 \). Hence, \( G'_0 \) has no path that is from \( a_0 \) to \( b_1 \) and internally disjoint from \( B' \). In particular, \( \alpha(A', B') \leq 1 \). \( \square \)

Since \( G^* \) is 6-connected, \( G \) has two disjoint \( A'-B' \) paths from \( p, q \in V(A'(x_1, x_2)) \) to \( p', q' \in V(B'[r_1, y_1]) \), respectively. We choose \( P, Q \) to first maximize \( A'[p, q] \), then minimize \( B'[b_1, p'] \cap B'[b_1, q'] \), and finally maximize \( B'[p', q'] \). By the symmetry between \( a_1 \) and \( a_2 \), we may relabel \( a_1, x_1, x_2, a_2 \) so that

- \( a_1, x_1, p, q, x_2, a_2 \) occur on \( A' \) in order, and \( b_1, r_1, p', q', y_1, b_2 \) occur on \( B' \) in order.

**Lemma 5.0.3** Any \( A'-B' \) path from \( B'[r_1, p'] \) to \( A'(x_1, x_2) \) must be disjoint from \( P, Q \), and end in \( A'(p, q) \). Moreover, if \( H' - y_1 \) contains a path from \( u \in A'[q, x_2] \) to \( y_2 \) and internally disjoint from \( A' \), then all \( A'-B' \) paths from \( A'(u, x_2) \) to \( B'[r_1, y_1] \) and internally disjoint from \( H' - y_1 \), are edges ending in \( \{r', y_1\} \).

**Proof.** First, assume \( S \) is an \( A'-B' \) path from \( s' \in V(B'[r_1, p']) \) to \( s \in V(A'(x_1, x_2)) \). Then \( V(S \cap (P \cup Q)) = \emptyset \); for otherwise, let \( v \in V(S \cap (P \cup Q)) \) with \( S[s', v] \) minimal then \( P' := S[s', v] \cup P[v, p] \) and \( Q \) (when \( v \in V(P) \)) or \( P \) and \( Q' := S[s', v] \cup Q[v, q] \) (when \( v \in V(Q) \)) contradict the choice of \( P, Q \). Hence, \( s \in A'(p, q) \) as otherwise \( S, P \) or \( S, Q \) contradict the choice of \( P, Q \).

Now let \( Y_2 \) be a path in \( H' - y_1 \) from \( u \in V(A'[q, x_2]) \) to \( y_2 \) and internally disjoint from \( A' \). We first see that \( G \) has no path from \( A'(u, x_2) \) to \( B'[r_1, y_1] - p' \). For, suppose not. Let \( S \) be an \( A'-B' \) path from \( s \in V(A'(u, x_2)) \) to \( s' \in V(B'[r_1, y_1] - p') \). Then \( V(S \cap P) \neq \emptyset \), or else, \( P, S \) contradict the choice of \( P, Q \). Since \( s' \neq p' \), \( S, P \) are contained in an \( A'-B' \) bridge. However, by \( u \in A'(p, s) \), the existence of \( Y_2 \) contradicts \((v)\) of Lemma 4.0.9.

Now let \( S \) be an arbitrary \( A'-B' \) path from \( s \in A'(u, x_2) \) to \( s' \in B'[r_1, y_1] \). Suppose \( S \) has length at least 2. Then \( S \) is contained in some \( A'-B' \) bridge \( N \) with feet \( n'_1, n'_2 \) and extreme hands \( n_1, n_2 \). Then \( n'_1, n'_2 \in \{p', y_1\} \). By \((v)\) of Lemma 4.0.9 and the existence of \( S \) and \( Y_2 \), \( A'[n_1, n_2] \subseteq A[u, x_2] \). Let \( h_1, h_2 \in A'[x_1, x_2] \), such that \( A'[n_1, n_2] \subseteq A'[h_1, h_2] \),
$H' - y_1$ does not contain a path from $A'(h_1, h_2)$ to $y_2$ and internally disjoint from $A'$, and subject to this, $A'[h_1, h_2]$ is maximal. Clearly, $A'(h_1, h_2) \subseteq A'(u, x_2)$, and for $i \in [2]$, $H' - y_1$ contains a path from $h_i$ to $y_2$ and internally disjoint from $A'$. By (v) of Lemma 4.0.9, \{h_1, h_2, p', y_1\} is a cut in $G^*$ separating $V(N)$ from \{a_0, a_1, a_2, b_1, b_2\}, a contradiction.

Thus, $S$ must be an edge. To complete the proof, we need to show $r' = p'$. For, suppose $r' \neq p'$. By (i), $R$ is disjoint from $P, Q$ with $r \in A'(p, q)$, and so $R, P, S, Y_2$ force a double cross in $A, B$, contradicting Lemma 2.0.7. □

Let $R = P$ if $r' = p'$, and if $r' \neq p'$ then by Lemma 5.0.3, $R$ is disjoint from $P, Q$ with $r \in A'(p, q)$. By Lemma 5.0.1, for $i \in [2]$, we let $P_{1,i}, P_{2,3-i}$ be disjoint paths from $x_1, x_2$ to $y_i, y_{3-i}$, respectively, in $H' - y_1 - A'(x_1, x_2)$.

We now use the structure inside $H'$ to derive further structure outside $H'$.

**Lemma 5.0.4**

(i) $G$ has no edge from $A'(x_2, a_2)$ to $B'(b_1, r_1)$ and no edge from $A'[a_1, x_1)$ to $B'[y_2, b_2)$.

(ii) $G$ has no edge from $b_1$ to $A'[a_1, x_1] \cup A'[x_2, a_2]$ and no edge from $b_2$ to $A'[x_2, a_2]$.

(iii) $r_1 = b_1$ implies $x_1 = a_1$, and $y_2 = b_2$ implies $x_2 = a_2$.

(iv) If $y_2 \neq b_2$ and $y_2$ is a cut vertex of $G_0'$ separating $b_2$ from \{a_0, b_1\}, then $N(b_2) = \{y_2, x_1\}, a_1 \neq x_1$, and $a_2 = x_2$.

**Proof.** By Lemma 4.0.7 and (iv) of Lemma 5.0.2, we may assume

1. when $b_1 \neq r_1$, $G_0' - B'(b_1, r') - B'[y_2, b_2]$ contains disjoint paths $B_1^*, A_0^*$ from $b_1, a_0$ to $q', y_1$, respectively.

2. $G$ has no edge from $A'(x_2, a_2)$ to $B'(b_1, r_1)$.

For, let $e = ab \in E(G)$ with $a \in A'(x_2, a_2)$ and $b \in B'(b_1, r_1)$. Then $b_1 \neq r_1$; so $B_1^*, A_0^*$ exist by (1). Now $A'[a_1, r] \cup R \cup B'[b, r'] \cup e \cup A'[a, a_2] \cup P_{1,1} \cup A_0^* \cup B_1^* \cup Q \cup A'[q, x_2] \cup P_{2,2} \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. □

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(3) G has no edge from $b_2$ to $A'[x_2, a_2]$.

For, let $e = ab_2 \in E(G)$ with $a \in A'[x_2, a_2]$. Then $a \neq a_2$ and let $e' = a_2b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'[y_2, b_2]$ to avoid the double cross $e, e', P_{1.2}, P_{2.1}$. Hence, $b' \in B'(b_1, r_1)$, contradicting (2). \quad \square

(4) G has no edge from $A'[a_1, x_1]$ to $B'[y_2, b_2]$.

Otherwise, let $e = ab \in E(G)$ with $a \in A'[a_1, x_1]$ and $b \in B'[y_2, b_2]$. Then $G$ has no edge from $b_2$ to $\{x_1, x_2\}$; as such an edge must be $b_2x_1$ by (3), which forms a double cross with $e, P_{1.1}$ and $P_{2.2}$, contradicting Lemma 2.0.7.

Hence, by Lemma 4.0.7 and (iv) of Lemma 5.0.2, $G'_0 = B'[b_1, r'] - B'[y_2, b]$ has disjoint paths $B_2, A_0$ from $b_2, a_0$ to $y_1, q'$, respectively. But then, $A'[a_1, a] \cup e \cup B'[y_2, b] \cup P_{2.2} \cup A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup A'[x_1, r] \cup P_{1.1} \cup B_2$ show that $\gamma$ is feasible, a contradiction. \quad \square

(5) (i)–(ii) hold.

For, suppose not. Then $G$ has an edge $e = b_1a$ with $a \in A'[a_1, x_1] \cup A'[x_2, a_2]$.

Suppose $a \in A'[a_1, x_1]$. Then $a \neq a_1$, and let $e' = a_1b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1)$ to avoid the double cross $e, e', P_{1.2}, P_{2.1}$. So $b' \in B'[y_2, b_2)$, contradicting (4).

Hence, $a \in A'[x_2, a_2]$. Then $a \neq a_2$, and let $e' = a_2b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1)$ to avoid the double cross $e, e', P_{1.1}, P_{2.2}$. Hence, $b' \in B'[y_2, b_2)$.

If $G$ has an edge $e_3$ from $b_2$ to $\{x_1, x_2\}$ then, by (3), it ends with $x_1$. So $a_1 \neq x_1$, and $G$ has an edge $e_4$ from $a_1$ to $B'(b_1, b_2)$. But now, $e, e', e_3, e_4$ force a double cross, a contradiction.

So $G$ has no edge from $b_2$ to $\{x_1, x_2\}$. Hence, by Lemma 4.0.7, $G'_0 = B'[b_1, r_1] - B'[y_2, b']$ has disjoint paths $B_2, A_0$ from $b_2, a_0$ to $y_1, q'$, respectively. But then, $A'[a_1, q]$ \quad
\[ P_{1,2} \cup B'[y_2, b'] \cup e' \cup Q \cup A_0 \text{ and } e \cup A'[x_2, a] \cup P_{2,1} \cup B_2 \text{ show that } \gamma \text{ is feasible, a contradiction.} \]

Since \( G^* \) is 6-connected, it follows from (2) and (4) that (iii) holds. It remains to prove (iv). So assume \( y_2 \neq b_2 \) and \( y_2 \) is a cut vertex of \( G'_0 \) separating \( b_2 \) form \( \{a_0, b_1\} \). Then \( \alpha(A', B') \leq 1. \)

Suppose \( B'(y_2, b_2) \neq \emptyset \). Then, since \( G^* \) is 6-connected, it follows from (4) that \( G \) has edges from \( B'(y_2, b_2) \) to distinct \( u_1, u_2 \in V(A'[x_2, a_2]) \), and we choose \( u_1, u_2 \) so that \( A'[u_1, u_2] \) is maximal. Now, by (2) and (3), \( \{u_1, u_2, y_2, b_2, x_1\} \) is a cut in \( G^* \) separating \( V(B'(y_2, b_2)) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

So \( B'(y_2, b_2) = \emptyset \). Then \( a_2 = x_2 \); for otherwise, since \( G^* \) is 6-connected, \( G \) has an edge from \( a_2 \) to \( B'(b_1, r_1) \), contradicting (2). We may assume that there exists \( e = b_2a \in E(G) \) with \( a \in A'(a_1, x_1) \); as otherwise, (iv) holds. Let \( e' = a_1b' \in E(G) \) with \( b' \in B'(b_1, b_2) \). Then \( b' \in B'(b_1, r_1) \) by (4); so \( b_1 \neq r_1 \), and \( B_1^*, A_0^* \) exist by (1). Now, by Lemma 3.0.1, we derive \( \alpha(A', B') = 2 \) with the following paths: the path \( e' \cup B'[b_1, b'] \) from \( a_1 \) to \( b_1 \), the path \( A'[a_1, a] \cup e \) from \( a_1 \) to \( b_2 \), the path \( B_1^* \cup Q \cup A'[x_1, q] \cup P_{1,2} \cup B'[y_2, b_2] \) from \( b_1 \) to \( b_2 \), and the path \( A_0^* \cup P_{2,1} \) from \( a_0 \) to \( a_2 \). This contradicts \( \alpha(A', B') \leq 1 \) as \( A', B' \) is a good frame.

Let \( H_0 \) denote the minimal union of blocks of \( H' - y_1 - A'[q, x_2] \) containing \( X_1 \), let \( W \) denote the path between \( x_1 \) and \( y_2 \), such that \( W \) is contained in the outer walk of \( H_0 \), and for any vertex \( v \in V(W - A') \), there exists a vertex \( u \in A'[q, x_2] \), such that \( u, v \) are incident with a finite face of \( H' - y_1 \), and let \( w_1 \in V(A' \cap W) \) with \( A'[x_1, w_1] \) maximal.

Next, we further study the structure inside \( H' \).

**Lemma 5.0.5**

(i) \( H_0 = H' - y_1 - A(w_1, x_2) \), and each vertex in \( W(w_1, y_2) \) has at most two neighbors on \( A'[q, x_2] \), inducing a subpath of \( A' \) with vertices at most two.

(ii) \( H' - \{y_1, y_2\} - A'(x_1, x_2) \) contains a path from \( x_1 \) to \( x_2 \).

**Proof.** Suppose (i) is not true. Then \( H' - y_1 \) has a nontrivial \((H_0 \cup A'[q, x_2])\)-bridge \( J \) which
has exactly one vertex in \( W(w_1, y_2) \) (by definition of \( H_0 \) and since \( G - A' \) is connected) or some vertex \( w \in V(W(w_1, y_2)) \) has two neighbors on \( A'[q, x_2] \) such that the subpath of \( A' \) between them has at least three vertices. In the first case, let \( w \in V(J \cap H_0) \) and \( u, v \in V(J \cap A') \) such that \( J \cap A' \subseteq A'[u, v] \); and in the second case, let \( u, v \) be the neighbors of \( w \) on \( A'[q, x_2] \) such that \( A'[u, v] \) is maximal. Then by Lemma 5.0.3, \( \{u, v, w, y_1, r'\} \) is a cut in \( G^* \), a contradiction.

Now suppose (ii) is not true. Then there exists \( v_0 \in V(A'(x_1, x_2)) \) such that \( y_2, v_0 \) are incident with a finite face of \( H' - y_1 \). We further choose \( v_0 \) so that \( A'[v_0, x_2] \) is minimal, and let \((L_1, L_2)\) be a separation in \( H' - y_1 \) such that \( V(L_1 \cap L_2) = \{y_2, v_0\}, x_1 \in V(L_1) \), and \( x_2 \in V(L_2) \).

By Lemma 5.0.1, for each \( j \in [2] \), \( H' - A'(x_1, x_2) \) contains disjoint paths from \( y_1, y_2 \) to \( x_j, x_{3-j} \), respectively. So for \( j \in [2] \), \( G[V(L_j) \cup \{y_1\}] - y_2 \) contains a path \( T_j \) from \( y_1 \) to \( x_j \) and internally disjoint from \( A' \).

We see that \( y_2, v_0 \) are not incident with a common finite face of \( H_0 \). For otherwise, \( v_0 \in A'(x_1, w_1), x_1 \neq w_1, \) and \( W[w_1, y_2] \subseteq L_2 \). Hence, \( T_1, W[w_1, y_2], P \) and \( Q \) are disjoint, which form a doublecross, a contradiction to Lemma 2.0.7.

Now, by the minimality of \( A'[v_0, x_2] \) and planarity of \( H' - y_1, v_0 \in A'[q, x_2] \). Therefore, by Lemma 5.0.3, \( \{v_0, x_2, r', y_1, y_2\} \) is a cut in \( G^* \) separating \( V(L_2) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Let \( w_2, \ldots, w_m \) be the vertices on \( W \) in order from \( x_1 \) to \( y_2 \) such that for \( i \in \{2, \ldots, m\} \), \( w_i \) has a neighbor on \( A'[q, x_2] \), and for \( i \in \{2, \ldots, m\} \), let \( u_i, v_i \in N(w_i \cap A') \), such that \( a_1, u_i, v_i, b_2 \) occur on \( A' \) in order with \( A'[u_i, v_i] \) maximal.

**Lemma 5.0.6** \( w_1 \neq x_1 \), and \( H_0 \) is 2-connected.

**Proof.** Suppose this is false. Let \( z \in V(H_0) \) such that \( z = x_1 \) (when \( x_1 = w_1 \)) or \( z \) is a cut vertex of \( H_0 \) and, subject to this, \( W[x_1, z] \) is maximal. Then \( V(W[z, y_2] \cap X_1) = \{z, y_2\} \). Note that \( z \in X_1[x_1, y_2] \) and \( w_m \in W(z, y_2) \).
Let \( k \) be minimum such that \( w_k \in W(z, y_2) \) and \( u \in N(w_k) \cap V(A'[q, x_2]) \) such that \( A'[q, u] \) is minimal. Moreover, let \( K \) denote the \( \{z, u\}\)-bridge of \( H - y_1 \) containing \( A'[u, x_2] \cup X_2 \), and let \( K^* := G[V(K) \cup \{y_1\}] \).

By (v) of Lemma 4.0.9 and by the existence of \( W[y_2, w_k] \cup w_ku \),

(1) no \( A'-B' \) bridge has one extreme hand in \( A'[x_1, u] \) and the other in \( A'(u, x_2) \).

Thus, since \( \{y_1, y_2, z, u, x_2\} \) is not a cut in \( G^* \) separating \( V(K) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), \( G \) has an \( A'-B' \) path from \( A'(u, x_2) \) to \( B'[r_1, y_1] \) and internally disjoint from \( H' \). By Lemma 5.0.3,

(2) all \( A'-B' \) paths from \( A'(u, x_2) \) to \( B'[r_1, y_1] \) and internally disjoint from \( H' \) are edges from \( A'(u, x_2) \) to \( \{r', y_1\} \).

So let \( e = ar' \in E(G) \) with \( a \in A'[u, x_2] \), and choose \( a \) such that \( A'[u, a] \) is minimal.

(3) Let \( L \) denote the path on the outer walk of \( K \) between \( y_2 \) and \( u \) not going through \( x_2 \), and let \( L_0 := L \cup A'[u, a] \). Then \( V(L_0 \cap X_2) = \{y_2\} \), and \( N(y_1) \cap V(K) \subseteq V(L_0) \).

First, suppose there exists \( v \in V(L_0 \cap X_2) \), such that \( v \neq y_2 \). Then \( \{v, y_1, u, x_2, r'\} \) is a cut in \( G^* \) separating \( V(A'(u, x_2)) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Now suppose there exist \( v \in N(y_1) \cap V(K) \) such that \( v \notin V(L_0) \). We claim that \( K^* - V(L_0) \) has a path \( Y_1 \) from \( y_1 \) to \( x_2 \). For otherwise, by the planar structure of \( K \), there exist \( c_1, c_2 \in V(L_0) \), such that \( c_1, c_2 \) are incident with a finite face of \( K \), and \( \{c_1, c_2\} \) is a 2-cut in \( K \) separating \( v \) from \( x_2 \). Thus, by (2) and the choice of \( a \), \( \{c_1, c_2, y_1, u, z\} \) is a cut in \( G^* \) separating \( v \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

If \( G \) has an \( A'-B' \) path \( T \) from \( A'(x_1, u) \) to \( B'(r', y_1) \) and internally disjoint from \( H' \), then \( T, e, L, Y_1 \) force a double cross, a contradiction. So \( T \) does not exist. Then \( u = q \) and, by (1), \( \{x_1, u, z, r'\} \) is a cut in \( G^* \) separating \( r \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. \( \Box \)

We will need the following claim.
(4) $G'_0$ contains a path $A'_0$ from $B'(r', y_1)$ to $a_0$ and internally disjoint from $B'$.

For otherwise, there exists $b'_1 \in V(B'[b_1, r'])$, such that $\{b'_1, y_1\}$ is a 2-cut in $G'_0$ separating $B'[b'_1, y_1]$ from $\{a_0, b_1, b_2\}$. Furthermore, $\{b'_1, y_1, y_2\}$ is a 3-cut in $G'_0$ separating $B'[b'_1, y_2]$ from $\{a_0, b_1, b_2\}$. We choose $b'_1$ so that $B'[b_1, b'_1]$ is minimal. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, $b'_1 = b_1$, and $\{b_1, y_1, y_2, b_2\}$ is a cut in $G^*$ separating $a_0$ from $\{a_1, a_2\}$, a contradiction. □

Let $y'_1, y''_1 \in V(L_0) \cap N(y_1)$ such that $a, y'_1, y''_1, y_2$ occur on $L_0$ in order and, subject to this, $L_0[y'_1, y''_1]$ is maximal.

(5) $y''_1 \in L_0[z, u]$.

For, otherwise, $y''_1 \in L_0(z, y_2)$. Then $y'_1 \notin L_0[z, y_2]$; otherwise, $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', u, z, y_1, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $G_2 = K^*$, and $(G_2, r', u, z, y_1, y_2, x_2)$ is planar, which contradicts Lemma 2.0.3.

We claim that $K - V(L_0[y'_1, a] \cup L_0[y_2, y''_1])$ contains a path $X'$ from $x_2$ to $z$. For otherwise, by (3) and the planar structure of $K$, there exist $c_1 \in V(L_0[y'_1, a])$ and $c_2 \in V(L_0[y_2, y''_1])$, such that $c_1, c_2$ are incident with a face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $x_2$ from $z$. If $c_1 \in A'[u, a]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in $G^*$ separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_1 \notin A'[u, a]$. Then $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', u, c_1, c_2, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[u, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', u, c_1, c_2, y_2, x_2)$ is planar. This contradicts Lemma 2.0.3.

Now, the following paths give a contradiction to (i) of Lemma 3.0.2: the path $A'[a_1, x_1] \cup X_1[x_1, z] \cup X' \cup A'[x_2, a_2]$ from $a_1$ to $a_2$, the path $B'[b_1, r'] \cup e \cup L_0[a, y'_1] \cup y'_1y_1 \cup y_1y''_1 \cup L_0[y''_1, y_2] \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, and the path $A'_0$ from $B'(r', y_1)$ to $a_0$. □

We claim that $y'_1 \in A'(u, a)$. For, otherwise, $y'_1, y''_1 \in L_0[z, u]$. Now, $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', u, y_1, z, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $G_2 = K^*$, and $(G_2, r', u, y_1, z, y_2, x_2)$ is planar. This contradicts Lemma 2.0.3.

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Moreover, $K - V(L_0[y'_1, a] \cup L_0[y_2, y''_1])$ contains a path $X'$ from $x_2$ to $u$. For otherwise, by (3) and the planar structure of $K$, there exist $c_1 \in V(L_0[y'_1, a])$ and $c_2 \in V(L_0[y_2, y''_1])$, such that $c_1, c_2$ are incident with a finite face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $x_2$ from $u$. If $c_2 \in L_0[y_2, z]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in $G^*$ separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_2 \notin L_0[y_2, z]$. Then $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', c_1, c_2, z, y_2, x_2\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(A'[c_1, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', c_1, c_2, z, y_2, x_2)$ is planar. This contradicts Lemma 2.0.3.

Hence, the following paths contradict (i) of Lemma 3.0.2: the path $A'[a_1, u] \cup X' \cup A'[x_2, a_2]$ from $a_1$ to $a_2$, the path $B'[b_1, r'] \cup e \cup L_0[a, y'_1] \cup y'_1y_1y_1y''_2 \cup L_0[y''_1, y_2] \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, and the path $\overline{A}'$ from $B'(r', y_1)$ to $a_0$.

**Lemma 5.0.7** Let $z_1, z_2 \in V(W)$ with $W[z_1, z_2]$ is maximal, such that $x_1, z_1, z_2, y_2$ occur on $W$ in order, and for each $i \in [2]$, $G[H_0 + y_1]$ has a path $Z_i$ from $y_1$ to $z_i$ and internally disjoint from $W$. Then, $N(y_1) \cap V(X_1[x_1, y_2]) = \emptyset$ and $Z_1 \cap (X_1 \cup X_2) = \emptyset$.

**Proof.** By Lemma 5.0.6, $w_1 \neq x_1$ and $H_0$ is 2-connected. So $V(X_1 \cap W) = \{x_1, y_2\}$.

If $N(y_1) \cap V(X_1[x_1, y_2]) \neq \emptyset$ or $Z_1 \cap X_1 \neq \emptyset$ then $Z_1 \cup X_1$ contains a path $S$ from $y_1$ to $x_1$ and disjoint from $W[w_1, y_2]$. Now $S, W[w_1, y_2], P$, and $Q$ force a double cross, contradicting Lemma 2.0.7. So $N(y_1) \cap V(X_1[x_1, y_2]) = \emptyset$ and $Z_1 \cap X_1 = \emptyset$.

Moreover, $Z_1 \cap X_2 = \emptyset$. For, otherwise, by the choice of $z_1$ and $Z_1$, it follows from the planarity of $H' - y_1$ that $z_1 \in V(X_2)$. But then, $H' - A'(x_1, x_2)$ contains no disjoint paths from $y_1, y_2$ to $x_1, x_2$, respectively. This contradicts Lemma 5.0.1.

**Lemma 5.0.8** $a_2 = x_2$, and if $y_2 \neq b_2$ then $y_1, y_2$ are cut vertices in $G'_0$ separating $b_2$ from $\{a_0, b_1\}, N(b_2) = \{y_2, x_1\}$, and $a_1 \neq x_1$. Moreover, one of the following holds:

(i) there exists a 2-cut $\{z'_1, z'_2\}$ in $H_0$ with $x_1, z'_1, z_1, z_2, z'_2, y_2$ on $W$ in order such that $W(z'_1, z'_2) \neq \emptyset$ and $z'_1, z'_2$ are incident with a finite face of $H_0$, or

(ii) $N(y_1) \cap V(H_0) \subseteq V(W[w_1, y_2])$ and, for any $i \in [m], w_i \notin W(z_1, z_2)$.
Proof. By Lemma 5.0.6, $w_1 \neq x_1$, and $H_0$ is 2-connected. If $y_2 = b_2$, then by (iii) of Lemma 5.0.4, we have $a_2 = x_2$.

Now assume $y_2 \neq b_2$. We claim that $G'_0$ has a 3-cut $\{a'_0, b'_1, y_2\}$ with $b'_1 \in B'[b_1, r_1]$, which separates $B'[b'_1, y_2]$ from $\{a_0, b_1, b_2\}$. For otherwise, by (iv) of Lemma 5.0.2, $G'_0 - B'[b_1, r'] - y_2$ contains disjoint paths $A_0, B_2$ from $a_0, b_2$ to $q', y_1$, respectively. Let $Y_1$ be a path in $Z_1 \cup W[z_1, w_1] \cup A'[w_1, r]$ from $y_1$ to $r$. Note that $r \notin A'[q, x_2]$ and, by Lemma 5.0.7, $Y_1 \cap (A'[q, x_2] \cup X_1 \cup X_2) = \emptyset$. Now, $A'[a_1, x_1] \cup X_1 \cup X_2 \cup A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup Y_1 \cup B_2$ show that $\gamma$ is feasible, a contradiction.

Thus, when $y_2 \neq b_2$, we may apply Lemma 4.0.7 (with $b'_2 = y_2$), and conclude that $b'_1 = b_1, a'_0 = a_0$, and $y_1, y_2$ are cut vertices in $G'_0$ separating $b_2$ from $\{a_0, b_1\}$. By (iv) of Lemma 5.0.4, we have $N_{G}(b_2) = \{y_2, x_1\}, a_1 \neq x_1$, and $a_2 = x_2$.

We now show (i) or (ii) holds. First, suppose $z_1 = z_2$. Then $N(y_1) \cap V(H_0) = \{z_1\}$; or else, there exists $v \in N(y_1) \cap V(H_0)$ with $v \neq z_1$, and $\{z_1, y_1\}$ is a cut in $G$ separating $v$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. Clearly, $z_1 \in V(W(w_1, y_2))$, and so (ii) holds.

So we may assume $z_1 \neq z_2$. Now suppose $W(z_1, z_2) \cap \{w_1, \ldots, w_m\} = \emptyset$. Then (ii) holds or there exists $v \in N(y_1) \cap V(H_0)$ such that $v \notin V(W)$. In the latter case, there exist $c_1, c_2 \in V(W(x_1, y_2))$, such that $\{c_1, c_2\}$ is a 2-cut in $H_0$ separating $v$ from $x_1$; since, otherwise, $H_0 - W(x_1, y_2)$ contains a path $T$ from $v$ to $x_1$, and $y_1 v \cup T, W[w_1, y_2], R, Q$ force a double cross, contradicting Lemma 2.0.7. Now, $\{y_1, c_1, c_2\}$ is a cut in $G^*$, a contradiction.

Hence, we may assume $W(z_1, z_2) \cap \{w_1, \ldots, w_m\} \neq \emptyset$. Now suppose (i) fails. Then by the planar structure of $H_0$, $H_0 - W(x_1, z_1) - W[z_2, y_2]$ contains a path $X'$ from $x_1$ to $W(z_1, z_2)$ and internally disjoint from $W$.

We claim that $X'$ must be disjoint from $Z_1, Z_2$. For otherwise, let $x^* \in V(X' \cap Z_j)$ for some $j \in [2]$. As $X', Z_1, Z_2$ are all internally disjoint from $W$, $Z_j[s_j, x^*] \cup X'[x^*, x_1]$ implies that $z_1 = x_1$, contradicting Lemma 5.0.7 that $V(Z_1 \cap (X_1 \cup X_2)) = \emptyset$.

We claim $w_1 \in W(z_1, z_2)$. For otherwise, $w_i \in W(z_1, z_2)$ for some $i \geq 2$. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, there exists a path $A'_0$ in $G'_0$ from $a_0$ to $B'(r', y_1)$ internally
disjoint from $B'$. Now $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup w_iv_i \cup A'[q, a_2] \cup Q \cup B'(r', y_1) \cup A^*_{y}$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction.

So $z_1 \in A'(x_1, w_1)$. Moreover, $r \notin A'(x_1, z_1)$; otherwise, $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup A'[w_1, a_2] \cup Q \cup B'(r', y_1) \cup A^*_{y}$ and $B'[b_1, r'] \cup R \cup A'[r, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. But now, $A'[a_1, z_1] \cup Z_1 \cup B'(r', y_1) \cup A^*_{y} \cup Q \cup A'[q, a_2]$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2] \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. □

**Lemma 5.0.9** Suppose (i) of Lemma 5.0.8 holds, and a 2-cut $\{z'_1, z'_2\}$ in $G^*_0$ is chosen with $W[z'_1, z'_2]$ maximal. Then $z'_1 \in A'[x_1, w_1]$.

**Proof.** For, suppose 2-cut $\{z'_1, z'_2\}$ is chosen with $W[z'_1, z'_2]$ maximal, and $z'_1 \notin A'[x_1, w_1]$.

By Lemma 5.0.5, we can define $u', u'', v', v''$, such that $u', u'' \in V(A'[q, x_2]), x_1, u', u'', x_2$ occur on $A'$ in order, $H' - y_1$ has edges from $u', u''$ to $v', v'' \in V(W(z'_1, z'_2))$, respectively, subject to this, $A'[u', u'']$ is maximal, and subject to this, $W[v', v'']$ is maximal. Obviously, there exists a separation $(K, K_0)$ in $H' - y_1$, such that $V(K \cap K_0) = \{u', u'', z'_1, z'_2\}$, $V(W[z'_1, z'_2] \cup A'[u', u'']) \subseteq V(K)$, and $V(W[x_1, z'_1] \cup X_1) \subseteq V(K_0)$. We also let $K^* := G[V(K) \cup \{y_1\}]$.

By (v) of Lemma 4.0.9 and by the existence of the paths from $u', u''$ to $y_2$, respectively, in $H' - y_1$,

1. there does not exist an $A'-B'$ bridge with extreme hands $n_1, n_2$, such that for some $v \in \{u', u''\}$, $n_1 \in A'[x_1, v]$ and $n_2 \in A'(v, x_2)$.

Now, since $\{y_1, z'_1, z'_2, u', u''\}$ is not a cut in $G$ separating $V(K)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, then, combined with (1), we may assume

2. $A'(u', u'') \neq \emptyset$, and $G$ has an $A'-B'$ path from $A'(u', u'')$ to $B'[r_1, y_1)$, internally disjoint from $H' - y_1$.

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By Lemma 5.0.3, we may further assume

(3) all $A'-B'$ paths from $A'(u', u'')$ to $B'[r_1, y_1]$, internally disjoint from $H' - y_1$, are edges from $A'(u', u'')$ to $\{r', y_1\}$.

Now, we let $e = ar' \in E(G)$ with $a \in A'[u', u'']$, such that $A'[u', a]$ is minimal.

(4) $G'_0$ contains a path $A'_0$ from $B'(r', y_1)$ to $a_0$, internally disjoint from $B'$.

For otherwise, there exists $b'_1 \in B'[b_1, r']$, such that $\{b'_1, y_1\}$ is a 2-cut in $G'_0$ separating $B'[b'_1, y_1]$ from $\{a_0, b_1, b_2\}$. Furthermore, $\{b'_1, y_1, y_2\}$ is a 3-cut in $G'_0$ separating $B'[b'_1, y_2]$ from $\{a_0, b_1, b_2\}$. By Lemma 4.0.7 and (iv) of Lemma 5.0.2, $b'_1 = b_1$, and $\{b_1, y_1, y_2, b_2\}$ is a cut in $G$ separating $a_0$ from $\{a_1, a_2\}$, a contradiction. □

Since $z'_1$ may not be chosen so that $z'_1 \in A'[x_1, w_1]$, then there does not exist a vertex $v \in A'[x_1, w_1]$, such that $v, z'_2$ are incident with a common finite face of $K_0$. Thus, we may assume

(5) $K_0 - V(A'[x_1, u'])$ contains a path $Y$ from $y_2$ to $z'_1$, internally disjoint from $A'$.

Let $L$ denote the path on the outer walk of $K$ from $z'_1$ to $u'$ without going through $u''$. Obviously, $z'_2 \notin V(L)$. Moreover, we let $L_0 := L \cup A'[u', a]$.

(6) $N(y_1) \cap V(K) \not\subseteq (V(L_0) \cup \{z'_2\})$.

For, suppose $N(y_1) \cap V(K) \subseteq (V(L_0) \cup \{z'_2\})$. Obviously, $V(L_0) \cap N(y_1) \neq \emptyset$; otherwise, $\{u', u'', z'_1, z'_2, r'\}$ is a cut in $G$ separating $V(K)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, we let $y'_1, y''_1 \in V(L_0) \cap N(y_1)$, such that $a, y'_1, y''_1, z'_1$ occur on $L_0$ in order, and $L_0[y'_1, y''_1]$ is maximal.

We first claim $y'_1 \in L_0(u', a]$. For otherwise, $y'_1, y''_1 \in V(L_0[z'_1, u'])$. Now, $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', u', y_1, z'_1, z'_2, u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(K) \subseteq V(G_2)$, and $(G_2, r', u', y_1, z'_1, z'_2, u'')$ is planar, which contradicts Lemma 2.0.3.
Now, we see that $y''_1 \in L_0[z'_1, u')$. For, suppose $y''_1 \notin L_0[z'_1, u')$. Then $y''_1 \in L_0[u', a]$ and $z'_2 \in N(y_1)$. Now, $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', y_1, u', z'_1, z'_2, u''\}$, \{a_0, a_1, a_2, b_1, b_2\} $\subseteq V(G_1)$, $V(K) \subseteq V(G_2)$, and $(G_2, r', y_1, u', z'_1, z'_2, u'')$ is planar, which contradicts Lemma 2.0.3.

Then we claim that $K - V(L_0[z'_1, y''_1] \cup L_0[y'_1, a]) \cup \{z'_2\}$ contains a path $X'$ from $u''$ to $u'$. For otherwise, by the planar structure of $K$, there exist $c_1 \in V(L_0[y'_1, a])$, $c_2 \in V(L_0[z'_1, y''_1]) \cup \{z'_2\}$, such that $c_1, c_2$ are incident with a common finite face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $u'$ from $u''$. By the existence of the path $u''v'' \cup W[v''v, v'] \cup v'w'$ from $u''$ to $w'$, we may assume $c_2 = v'$. Moreover, $v' \neq v''$; otherwise, $\{v', u', u'', r', y_1\}$ is a cut in $G$ separating $V(A'(u', u''))$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. But then, as $G\ast$ is 6-connected, $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', c_1, v', z'_1, z'_2, u''\}$, \{a_0, a_1, a_2, b_1, b_2\} $\subseteq V(G_1)$, $V(A'[c_1, u'']) \cup \{v''\} \subseteq V(G_2)$, and $(G_2, r', c_1, v', z'_1, z'_2, u'')$ is planar, which contradicts Lemma 2.0.3.

Now, the path $A'[a_1, u'] \cup X' \cup A'[u'', a_2]$ from $a_1$ to $a_2$, the path $B'[b_1, r'] \cup c \cup L_0[a, y'_1] \cup y'_1y_1 \cup y_1y''_1 \cup L_0[y''_1, z'_1] \cup Y \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, and the path $A''_0$ from $B'(r', y_1)$ to $a_0$ contradict (i) of Lemma 3.0.2. □

(7) $K\ast - V(L_0) \cup \{z'_2\}$ contains a path $Y_1$ from $y_1$ to $u''$.

For, suppose (7) fails. By (6), there exists $v \in N(y_1) \cap V(K)$, such that $v \notin V(L_0) \cup \{z'_2\}$.

Since $K\ast - V(L_0) \cup \{z'_2\}$ contains no path from $y_1$ to $u''$, then by the planar structure of $K$, there exist $c_1, c_2 \in V(L_0) \cup \{z''_2\}$, such that $c_1, c_2$ are incident with a common finite face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $v$ from $u''$. Thus, combined with (3) and the choice of $a$, $\{c_1, c_2, y_1, u', z'_1\}$ is a cut in $G$ separating $v$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(8) $G$ has no $A'-B'$ path from $A'[a_1, u']$ to $B'(r', y_1)$, internally disjoint from $H' - y_1$.

For, suppose $G$ has an $A'-B'$ path $T$ from $A'[a_1, u']$ to $B'(r', y_1)$, internally disjoint from $H' - y_1$. Then $T, e, Y \cup L, Y_1$ force a doublecross, a contradiction. □
(9) \( b_1 = r_1 = r' \).

We may assume \( b_1 = r_1 \) and so \( a_1 = x_1 \) by (iii) of Lemma 5.0.4. For, suppose \( b_1 \neq r_1 \). By Lemma 4.0.7 and (iv) of Lemma 5.0.2, \( G'_0 - r' - B'[y_2, b_2] \) contains disjoint paths \( B_1, A_0 \) from \( b_1, a_0 \) to \( q', y_1 \), respectively. Now, \( A'[a_1, r] \cup R \cup e \cup A'[a_2] \cup Y_1 \cup A_0 \) and \( B_1 \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2] \) show that \( \gamma \) is feasible, a contradiction.

We may assume \( r_1 = r' \). For, suppose \( r_1 \neq r' \). By (iii) of Lemma 5.0.2, there exists an \( A'-B' \) bridge \( M_4 \) with feet \( l_4', r_4' \), such that \( R \) is internally disjoint from \( M_4 \), and \( r' \in B'(l_4', r_4') \). Let \( P^* \) be the path from \( l_4' \) to \( r_4' \) in \( M_4 \), internally disjoint from \( A', B' \), and let \( A'_0 \) be the path from \( a_0 \) to \( y_1 \) in \( G'_0 \), internally disjoint from \( B' \), then \( A'[a_1, r] \cup R \cup e \cup A'[a_2] \cup Y_1 \cup A'_0 \) and \( B'[b_1, l_4'] \cup P^* \cup B'[r_4', q'] \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2] \) show that \( \gamma \) is feasible, a contradiction. \( \square \)

Now, by (8), (9), Lemma 5.0.8, and Lemma 5.0.3, \( \{b_1, u', a_2, y_1, b_2\} \) is a cut in \( G \) separating \( a_0 \) from \( a_1 \), a contradiction. \( \square \)

**Lemma 5.0.10**

(i) \( \alpha(A', B') = 1 \), and \( y_1 \) is a cut vertex in \( G'_0 \) separating \( b_2 \) from \( \{a_0, b_1\} \);

(ii) Let \( A'_0 \) be the path from \( y_1 \) to \( a_0 \) in \( G'_0 \), internally disjoint from \( B' \) and on the boundary of \( G'_0 \), then \( G'_0 - B'(b_1, r') - A'_0 \) contains a path \( B'_1 \) from \( b_1 \) to \( q' \).

**Proof.** We may assume \( H' - \{y_1, y_2\} - Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2) \) contains a path \( X_0 \) from \( x_1 \) to \( x_2 \). For otherwise, by the planar structure of \( H' - y_1 \), there exists a vertex \( v \in V(Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2)) \), such that \( y_2, v \) are incident with a common finite face of \( H_0 \). By Lemma 5.0.5, \( v \notin A'(x_1, x_2) \), and so \( v \in V(Z_1 \cup W[z_1, w_1]) \). Now, we claim that there exists \( c \in W[x_1, z_1] \), such that \( \{y_2, c\} \) is a cut in \( H_0 \) separating \( W(c, y_2) \) from \( x_1 \). For otherwise, \( v \notin W[z_1, w_1] \). So \( v \in Z_1[s_1, z_1] \), and (i) of Lemma 5.0.8 holds with 2-cut \( \{z_1', z_1''\} \). But then, \( z_1', v, y_2 \) are incident with a common finite face of \( H_0 \), and \( \{y_2, z_1'\} \) is a 2-cut, which still leads to our claim. Thus, our claim is true. Now, the existence of
\{y_2, c\} show that (i) of Lemma 5.0.8 holds. By Lemma 5.0.9, \{y_2, c\} may be chosen with 
c \in A'[x_1, w_1], a contradiction to Lemma 5.0.5.

We now prove \(y_1\) is a cut vertex in \(G'_0\) separating \(b_2\) from \(\{a_0, b_1\}\). For, suppose not. Then by Lemma 5.0.8, we may assume \(y_2 = b_2\).

Moreover, we may assume \(G'_0 - B'[b_1, r'] - B'(y_1, b_2)\) contains disjoint paths \(A_0, B_2\) frpm \(a_0, b_2\) to \(q', y_1\), respectively. For otherwise, by planar structure of \(G'_0\), there exists a 3-cut \(\{a'_0, b'_1, b'_2\}\) with \(b'_1 \in B'[b_1, r']\) and \(b'_2 \in B'(y_1, b_2)\), which separates \(B'(b'_1, b'_2)\) from \(\{a_0, b_1, b_2\}\). Since \(y_1, b_2, b'_2\) are incident with a common finite face of \(G'_0\), then \(a'_0, b_2\) are incident with a common finite face of \(G'_0\), and so \(\{b'_1, a'_0, b_2\}\) is a 3-cut in \(G'_0\). Moreover, since \(y_1\) is not a cut vertex in \(G'_0\), then \(a'_0 = a_0\). But now, by (iv) of Lemma 5.0.2, \(b'_1 \notin B'(r_1, r')\), and therefore, \(b'_1 \in B'[b_1, r_1]\). Now, by Lemma 4.0.7, \(b'_1 = b_1\). Then \(\{b_1, b_2, a'_0\}\) is a cut in \(G\) separating \(a_0\) from \(\{a_1, a_2\}\), a contradiction.

Now, by the existence of \(A_0, B_2, A'[a_1, x_1] \cup X_0 \cup A'[q, a_2] \cup Q \cup A_0\) and \(B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z'_1 \cup B_2\) show that \(\gamma\) is feasible, a contradiction. Thus, \(y_1\) is a cut vertex in \(G'_0\), and \(\alpha(A', B') \leq 1\).

Next, we show that \(\alpha(A', B') = 1\). We let \(A^*_0\) be the path from \(a_0\) to \(y_1\) in \(G'_0\), internally disjoint from \(B'\). When \(y_2 = b_2\), we let \(B^* := A'[a_1, x_1] \cup X_1\); when \(y_2 \neq b_2\), by Lemma 5.0.8, \(x_1b_2 \in E(G)\), and we let \(B^* := A'[a_1, x_1] \cup x_1b_2\). Then combined with Lemma 3.0.1, the path \(A^*_0 \cup B'[q', y_1] \cup Q \cup A'[q, a_2]\) from \(a_0\) to \(a_2\), the path \(B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2] \cup B'[y_2, b_2]\) from \(b_1\) to \(b_2\), and the path \(B^*\) from \(a_1\) to \(b_2\) show that \(\alpha(A', B') = 1\).

Finally, we prove (ii) holds. For otherwise, by planar structure of \(G'_0\), there exists a 2-cut \(\{a'_0, b'_1\}\) with \(a'_0 \in A'_0\) and \(b'_1 \in B'(b_1, r']\), which separates \(b_1\) from \(q'\). Since \(y_1\) is a cut vertex of \(G'_0\), then \(\{a'_0, b'_1, b_2\}\) is a 3-cut in \(G'_0\) separating \(B'[b'_1, b_2]\) from \(\{a_0, b_1, b_2\}\).

By Lemma 4.0.7, \(b'_1 \notin B'(b_1, r_1)\), and so \(b'_1 \in (r_1, r']\). But, by (iv) of Lemma 5.0.2, \(a'_0 = a_0\), which implies that \(G'_0\) has no path from \(a_0\) to \(b_1\), internally disjoint from \(B'\), and so \(\alpha(A', B') = 0\), a contradiction. \(\square\)
Lemma 5.0.11 Suppose (i) of Lemma 5.0.8 does not hold and (ii) of Lemma 5.0.8 holds, then \( N(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2]) \).

Proof. Since \( z_1 \notin V(X_2) \) (by Lemma 5.0.7), then \( z_1 \notin W[w_m, y_2] \). So, by (ii) of Lemma 5.0.8, we may assume that there exists \( j \in [m - 1] \), such that \( z_1, z_2 \in W[w_j, w_{j+1}] \) with \( z_2 \in W(w_j, w_{j+1}) \). We may also assume \( z_2 \notin W[w_1, w_2] \) and \( j \neq 1 \); otherwise, \( N(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2]) \).

Since (i) of Lemma 5.0.8 does not hold and \( z_2 \notin W[w_1, w_2] \), then we may assume \( H_0 - W[x_1, w_1] - W[z_2, w_m] \) contains a path \( Y_2 \) from \( y_2 \) to \( w_2 \).

We may assume \( b_2 = y_2 \). For, suppose \( b_2 \neq y_2 \). Then by Lemma 5.0.8, \( G \) has an edge from \( b_2 \) to \( x_1 \), and \( a_1 \neq x_1 \). Let \( e = a_1b \in E(G) \) with \( b \in B'(b_1, r_1) \). Now, combined with Lemma 3.0.1, the path \( A'_0 \cup y_1z_2 \cup W[z_2, w_m] \cup w_mx_2 \) from \( a_0 \) to \( a_2 \), the path \( B'_1 \cup Q \cup A'[w_1, q] \cup W[w_1, w_2] \cup Y_2 \cup B'[y_2, b_2] \) from \( b_1 \) to \( b_2 \), the path \( e \cup B'[b_1, b] \) from \( a_1 \) to \( b_1 \), and the path \( A'[a_1, x_1] \cup x_1b_2 \) from \( a_1 \) to \( b_2 \) show that \( \alpha(A', B') = 2 \), a contradiction to (i) of Lemma 5.0.10.

Now, we distinguish two cases.

Case 1. \( w_2 = x_2 \).

(1.1) There does not exist a cross \( C, D \) from \( c, d \in A'[x_1, x_2] \) to \( c', d' \in B'[b_1, y_1] \), such that, \( c \in A'[a_1, w_1] \), \( a_1, c, d, a_2 \) occur on \( A' \) in order, and \( C, D \) are internally disjoint from \( A', B', H' \).

For otherwise, such a cross \( C, D \), together with the path \( y_1z_2 \cup W[z_2, w_m] \cup w_mx_2 \) from \( y_1 \) to \( x_2 \) and the path \( Y_2 \cup W[w_2, w_1] \) from \( y_2 \) to \( w_1 \), forces a doublecross. \( \square \)

(1.2) \( G \) has an \( A'-B' \) path \( T \) from \( t \in A'[a_1, w_1] \) to \( t' \in B'[b_1, y_1] \), internally disjoint from \( H' \).

For, suppose not, then \( \{a_1, w_1, x_2, y_1, y_2\} \) is a 5-cut in \( G \) separating \( V(H_0) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. \( \square \)
We further choose $T$ so that $B'[b_1, t']$ is minimal, and subject to this, $A'[a_1, t]$ is minimal.

(1.3) $t' \in B'[b_1, r']$, $V(T \cap Q) = \emptyset$, and $G$ has no $A'$-$B'$ path from $A'[a_1, t]$ to $B'[b_1, y_1]$, internally disjoint from $H'$.

We first prove $t' \in B'[b_1, r']$. For, suppose $t' \in B'(r', y_1)$. Then by the choice of $T$, we may assume $T, R$ are disjoint, and $r \in A'[w_1, q)$. But then, $T, R$ form a cross, contradicting (1.1).

We may assume $V(T \cap Q) = \emptyset$; otherwise, $T, Q$ are contained in some $A'$-$B'$ bridge, then by $w_1 \in A'(t, q)$, the path from $w_1$ to $y_2$ in $H' - y_1$, contradicting (v) of Lemma 4.0.9.

Finally, suppose $G$ has an $A'$-$B'$ path $S$ from $s \in A'[a_1, t]$ to $s' \in B'[b_1, y_1]$, internally disjoint from $H'$. Then by the choice of $T$, we have $S, T$ are disjoint, and $s \in B'(t', y_1)$. But then, $T, S$ form a cross, contradicting (1.1). □

(1.4) $H_0 - A'[x_1, t] \cup X_1[x_1, y_2] \cup W[z_1, w_3]$ contains a path $Y_2$ from $y_2$ to $w_1$.

For otherwise, by the planar structure of $H_0$, there exist $c_1 \in W[z_1, w_3]$ and $c_2 \in A'[x_1, t] \cup X_1[x_1, y_2]$, such that $\{c_1, c_2\}$ is a cut in $H_0$ separating $y_2$ from $w_1$. We notice that $j < m$ and $z_1 \notin V(X_2)$, and so $z_1 \in W[w_j, w_m)$. We may assume $c_2 \notin X_1[x_1, y_2]$; otherwise, $\{c_1, c_2, y_1, y_2, x_2\}$ is a cut in $G$ separating $w_m$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So, $t \in A'(x_1, w_1)$ and $c_2 \in A'(x_1, t)$. But then, by (1.3), $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{x_1, y_2, x_2, y_1, c_1, c_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(X_1 \cup X_2) \subseteq V(G_2)$, and $(G_2, x_1, y_2, x_2, y_1, c_1, c_2)$ is planar, which contradicts Lemma 2.0.3. □

Now, combined with Lemma 3.0.1, the path $A_6 \cup y_1 z_1 \cup W[z_1, w_j] \cup w_j x_2$ from $a_0$ to $a_2$, the path $B'_1 \cup Q \cup A'[w_1, q] \cup Y'_2$ from $b_1$ to $b_2$, the path $A'[a_1, t] \cup T \cup B'[b_1, t']$ from $a_1$ to $b_1$, and the path $A'[a_1, x_1] \cup X_1$ from $a_1$ to $b_2$ show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

Case 2. $u_2 \neq x_2$.

(2.1) $G$ has no $A'$-$B'$ path from $a_2$ to $B'(b_1, r')$.  

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For, suppose \( G \) has an \( A'-B' \) path \( S \) from \( a_2 \) to \( s' \in B'(b_1, r') \). Then, combined with (ii) of Lemma 5.0.10, \( A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup a_2w_m \cup W[w_m, z_2] \cup z_2y_1 \cup A'_0 \) and \( B'_1 \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y_2 \) show that \( \gamma \) is feasible, a contradiction. \( \Box \)

(2.2) There does not exist a cross \( C, D \) from \( c, d \in A'[x_1, x_2] \) to \( c', d' \in B'[b_1, y_1] \), such that \( a_1, c, d, a_2 \) occur on \( A' \) in order, and \( C, D \) are internally disjoint from \( A', B', H' \).

For, suppose such a cross exists. We claim that \( c \notin A'[a_1, u_2] \); otherwise, such a cross \( C, D \), together with the path \( y_1z_2 \cup W[z_2, w_m] \cup w_mx_2 \) from \( y_1 \) to \( x_2 \) and the path \( Y_2 \cup w_2u_2 \) from \( y_2 \) to \( u_2 \), forces a doublecross.

So, \( d \in A'(u_2, x_2) \). Then, by Lemma 5.0.3, \( D \) is an edge with \( d' = r' \). Moreover, by (2.1), \( b_1 = r' \).

Now, we may assume \( G \) has no \( A'-B' \) path from \( A'[a_1, u_2] \) to \( B'(b_1, y_1) \), internally disjoint from \( H' \); otherwise, such a path together with \( D \) forms a cross, contradicting our claim that \( c \notin A'[a_1, u_2] \). But then, combined with Lemma 5.0.3, \( \{b_1, b_2, y_1, u_2, a_2\} \) is a cut in \( G \) separating \( a_1 \) from \( a_0 \), a contradiction. \( \Box \)

(2.3) \( H_0 - A'(x_1, w_1) - W[z_2, y_2] \) has a path \( X' \) from \( x_1 \) to \( w_j \).

For otherwise, by planarity of \( H_0 \), there exist \( c_1 \in A'(x_1, w_1) \) and \( c_2 \in W[z_2, y_2] \), such that \( c_1, c_2 \) are incident with a common finite face of \( H_0 \), and \( \{c_1, c_2\} \) is a cut in \( H_0 \) separating \( x_1 \) from \( w_j \). But then, (i) of Lemma 5.0.8 holds, a contradiction. \( \Box \)

(2.4) \( H_0 - A'[x_1, w_1] \cup X_1[x_1, y_2] \cup W[z_2, w_m] \) contains a path \( Y_2^* \) from \( y_2 \) to \( w_2 \).

For otherwise, by planarity of \( H_0 \), there exist \( c_1 \in W[z_2, w_m] \) and \( c_2 \in A'[x_1, w_1] \cup X_1[x_1, y_2] \), such that \( c_1, c_2 \) are incident with a common finite face of \( H_0 \). Clearly, \( c_2 \notin A'[x_1, w_1] \); otherwise, (i) of Lemma 5.0.8 holds, a contradiction. So \( c_2 \in X_1[x_1, y_2] \).

Now, let \( w_i \in W(c_1, y_2) \) such that \( i \) is minimum. Then we may assume \( G \) has an \( A'-B' \) path \( S \) from \( s \in A'(u_i, x_2) \) to \( s' \in B'[b_1, y_1] \), internally disjoint from \( H' \); otherwise, \( \{u_i, c_1, c_2, y_2, x_2\} \) is a cut in \( G \) separating \( w_m \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

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By Lemma 5.0.3, \(S\) is an edge with \(s' \in \{r', y_1\}\). Now, if \(s' = y_1\), then \(A'[a_1, w_1] \cup W[w_1, z_1] \cup z_1y_1 \cup A'_0 \cup s's \cup A'[s, a_2]\) and \(B'[b_1, q'] \cup Q \cup A'[q, u_1] \cup u_1w_1 \cup W[w_1, y_2]\) show that \(\gamma\) is feasible, a contradiction. So \(s' = r'\). But then, \(S, Q\) form a cross, contradicting (2.2). □

(2.5) \(z_1, x_2\) are incident with a common finite face of \(H' - y_1\).

For otherwise, we may assume there exists some \(k \in \{j + 1, \cdots, m\}\), such that \(G\) has an edge from \(w_k\) to \(u_k \in A'[u_2, x_2]\). We further choose \(k\) so that \(k\) is minimum, and so \(k = j + 1\) or \(k = j + 2\).

Now, we claim that \(G\) has no \(A'-B'\) path from \(a_2\) to \(B'[b_1, y_1]\), internally disjoint from \(H'\). For, suppose \(G\) has an \(A'-B'\) path \(S\) from \(a_2\) to \(s' \in B'[b_1, r']\). By (2.1), \(s' \notin B'(b_1, r')\). We may also assume \(s' \notin B'(r', y_1)\); otherwise, \(S\) together with \(R, u_kw_k \cup W[w_k, y_2]\), and \(X' \cup W[w_j, z_1] \cup z_1y_1\) forces a doublecross. So \(s' = b_1\). Moreover, since \(G\) has no edge from \(a_2\) to \(b_1\), then \(S\) is not an edge, and so \(s' = r_1 = b_1\) and \(S\) is contained in an \(A'-B'\) bridge \(N\) with extreme hands \(n_1, n_2\) and feet \(n'_1, n'_2\). Since \(s' = b_1\), we have \(n'_1 = n'_2 = b_1\). By Lemma 5.0.3, \(V(N \cap A'(u_2, x_2)) = \emptyset\). By (v) of Lemma 4.0.9, \(n_1 \notin A'(a_1, u_2)\), and so \(n_1 = u_2\). But then, \(\{n_1, n_2, b_1\}\) is a cut in \(G\) separating \(V(N)\) from \(\{a_0, a_1, a_2, b_1, b_2\}\), a contradiction.

Now, since the degree of \(a_2\) in \(G\) is at least 4, then we may assume \(G\) has an edge from \(a_2\) to \(w \in W[w_k, w_m]\). But then, combined with Lemma 3.0.1, the path \(A'[a_1, r] \cup R \cup B'[b_1, r']\) from \(a_1\) to \(b_1\), the path \(A'[a_1, x_1] \cup X_1\) from \(a_1\) to \(b_2\), the path \(B'_1 \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y'_2\) from \(b_1\) to \(b_2\), and the path \(a_2w \cup W[w, z_2] \cup z_2y_1 \cup A'_0\) from \(a_2\) to \(a_0\) show that \(\alpha(A', B') = 2\), a contradiction to (i) of Lemma 5.0.10. □

(2.6) \(G\) has two disjoint \(A'-B'\) paths from \(A'(x_1, v_j)\) to \(B'[b_1, y_1]\), internally disjoint from \(H'\).

For otherwise, there exists a vertex \(v \in V(G)\), such that \(G - v\) does not contain any \(A'-B'\) paths from \(A'(x_1, v_j)\) to \(B'[b_1, y_1]\), internally disjoint from \(H'\). But then, combined with
(2.6) $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{v, x_1, y_2, x_2, u, v_j\}$ with $u = y_1$ (when $z_1 \neq z_2$) or $u = z_1$ (when $z_1 = z_2$), $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and $V(A'[x_1, v_j] \cup X_1) \subseteq V(G_2)$.

Now, we claim that $(G_2, v, x_1, y_2, x_2, u, v_j)$ is planar, and so Lemma 2.0.3 applies. Obviously, when $v \in A'$, this claim is true. So we may assume $v \notin A'$. Furthermore, if all the $A'-B'$ paths from $v$ to $A'$ are edges, then our claim is still true. Therefore, we may assume there exists some $A'-B'$ bridge $N$ with feet $n'_1, n'_2$ and extreme hands $n_1, n_2$, such that $v \in N$ and $N$ contains a path $P^*$ from $v$ to $A'[n_1, n_2]$, which is not an edge and internally disjoint from $A'$. By (v) of Lemma 4.0.9, $H' - y_1$ does not contain a path from $A'(n_1, n_2)$ to $y_2$, internally disjoint from $A'$. Hence, $\{n_1, n_2, v\}$ is a cut in $G$ separating $V(P^*)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

Now, we let $T_1, T_2$ be disjoint $A'-B'$ paths from $t_1, t_2 \in A'(x_1, u_j)$ to $t'_1, t'_2 \in B'[b_1, y_1]$, respectively, internally disjoint from $H'$, such that $a_1, t_1, t_2, a_2$ occur on $A'$ in order, subject to this, $B'[t'_1, t'_2]$ is maximal, and subject to this, $A'[t_1, t_2]$ is maximal. By (2.2), $b_1, b'_1, t'_2, b_2$ occur on $B'$ in order.

(2.7) $t'_1 \in B'[b_1, r']$.

For otherwise, $t'_1 \notin B'[r', y_1]$. We may first assume $R$ is internally disjoint from $T_1, T_2$. For otherwise, let $v \in V(R \cap (T_1 \cup T_2))$, such that $R[r', v]$ is minimal. If $v \in V(T_1)$, then $R[r', v] \cup T_1[v, t_1]; T_2$ contradict the choice of $T_1, T_2$; if $v \in V(T_2)$, then $T_1, R[r', v] \cup T_2[v, t_2]$ form a cross, contradicting (2.2).

Now, if $r \in A'[a_1, t_1]$, then $R, T_2$ contradict the choice of $T_1, T_2$; if $r \in A'(t_1, q)$, then $R, T_1$ form a cross, contradicting (2.2). □

(2.8) There exist $c_1, c_2 \in V(G'_0)$, such that $c_1 \in B'[b_1, t'_1]$, $c_2 \in B'[t'_2, y_1]$, and $c_1, c_2$ are incident with a common finite face of $G'_0$.

In fact, due to the existence of the path $A'[a_1, x_1] \cup X' \cup w_jv_j \cup A'[v_j, a_2]$ from $a_1$ to $a_2$ and the path $B'[b_1, t'_1] \cup T_1 \cup A'[t_1, t_2] \cup T_2 \cup B'[t'_2, y_1] \cup y_1z_2 \cup W[z_2, y_2]$ from $b_1$ to $b_2$,
$G'_0$ contains no path from $a_0$ to $B'(t'_1, t'_2)$, internally disjoint from $B'$ (by Lemma 3.0.2).

Hence, (2.8) holds. \(\square\)

Now, we further choose $c_1, c_2$ so that $B'[c_1, c_2]$ is maximal.

(2.9) $G'_0 - B'(b_1, c_1) \cup B'[c_2, y_1] \cup A'_0$ contains a path $B'_0$ from $b_1$ to $c_1$.

For otherwise, $V(B'(b_1, c_1)) \neq \emptyset$, and by planarity of $G'_0$, we may assume there exist $b'_1 \in B'(b_1, c_1)$ and $a'_0 \in B'[c_2, y_1] \cup A'_0$, such that $b'_1, a'_0$ are incident with a common finite face of $G'_0$. Now, if $a'_0 \in B'[c_2, y_1]$, then $b'_1, a'_0$ contradict the choice of $c_1, c_2$; if $a'_0 \in A'_0$, then $\{b'_1, a'_0, b_2\}$ is a 3-cut in $G'_0$, contradicting Lemma 4.0.7. \(\square\)

(2.10) $G'_0 - B'(b_1, c_2) \cup B'(c_2, y_1) \cup A'_0$ contains a path $B''_0$ from $b_1$ to $c_2$.

For otherwise, by planarity of $G'_0$, we may assume there exist $b'_1 \in B'(b_1, c_2)$ and $a'_0 \in B'(c_2, y_1) \cup A'_0$, such that $b'_1, a'_0$ are incident with a common finite face of $G'_0$. Now, if $a'_0 \in B'(c_2, y_1)$, then $b'_1, a'_0$ or $c_1, a'_0$ contradict the choice of $c_1, c_2$. So $a'_0 \in A'_0$. We may further assume $b_1 = c_1$ and $b'_1 \in B'(c_1, c_2)$; otherwise, $\{b'_1, a'_0, b_2\}$ or $\{c_1, a'_0, b_2\}$ is a 3-cut in $G'_0$, contradicting Lemma 4.0.7. But now, since $a_0, b_1, b'_1, c_2$ are incident with a common finite face of $G'_0$, then $\alpha(A', B') = 0$, a contradiction to (i) of Lemma 5.0.10. \(\square\)

(2.11) $G$ has no $A'-B'$ path from $B'(b_1, c_1)$ to $A'$.

For otherwise, since $c_1 \in B'[b_1, t'_1]$ and $t'_1 \in B'[b_1, r_1]$, then $c_1 \in B'[b_1, r_1]$, and so such an $A'-B'$ path from $B'(b_1, c_1)$ to $A'$ should be an edge $e$ from $b \in B'(b_1, c_1)$ to $a \in A'[a_1, x_1] \cup \{a_2\}$. By (2.2), $a \neq a_2$, and so $a \in A'[a_1, x_1]$.

But then, combined with Lemma 3.0.1, the path $A'[a_1, a] \cup e \cup B'[b_1, b]$ from $a_1$ to $b_1$, the path $A'[a_1, x_1] \cup X_1$ from $a_1$ to $b_2$, the path $A'_0 \cup B'[q', y_1] \cup Q \cup A'[a_2]$ from $a_0$ to $a_2$, and the path $B'_0 \cup B'[c_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2]$ from $b_1$ to $b_2$ show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. \(\square\)

(2.12) $G$ has an $A'$-$B'$ path $T$ from $t' \in B'(c_2, y_1)$ to $t \in A'[x_1, x_2]$. 61
For otherwise, combined with (2.11), \( \{b_1, c_1, c_2, y_1, b_2\} \) is a cut in \( G \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction. \( \square \)

Now, we choose \( T \) so that \( A'[t, a_2] \) is minimal.

\[(2.13) \quad t \neq a_2.\]

For otherwise, combined with Lemma 3.0.1, the path \( A'[a_1, t_1] \cup T_1 \cup B'[b_1, t'_1] \) from \( a_1 \) to \( b_1 \), the path \( A'[a_1, x_1] \cup X_1 \) from \( a_1 \) to \( b_2 \), the path \( T \cup B'[t', y_1] \cup A'_0 \) from \( a_2 \) to \( a_0 \), and the path \( B'' \cup B'[t_2, c_2] \cup T_2 \cup A'[t_2, u_2] \cup u_2w_2 \cup W[w_2, y_2] \) from \( b_1 \) to \( b_2 \) show that \( \alpha(A', B') = 2 \), a contradiction to (i) of Lemma 5.0.10. \( \square \)

\[(2.14) \quad T \text{ is internally disjoint from } T_1, T_2, \text{ and } t = u_2 = v_j.\]

First, we may assume \( T \) is internally disjoint from \( T_1, T_2 \). For otherwise, let \( v \in V(T \cap (T_1 \cup T_2)) \), such that \( T[v, t'] \) is minimal. Now, if \( v \in T_1 \), then \( T_1[t_1, v] \cup T[v, t'], T_2 \) form a cross, contradicting (2.2); if \( v \in T_2 \), then \( T_1, T_2[t_2, v] \cup T[v, t'] \) contradict the choice of \( T_1, T_2 \).

Now, by (2.2), we may assume \( t \in A'[t_2, a_2] \). By the choice of \( T_1, T_2 \), we may further assume \( t \notin A'[t_2, v_j] \). Finally, by Lemma 5.0.3, we have \( t \notin A'(u_2, a_2) \), and so \( t = u_2 = v_j \). \( \square \)

\[(2.15) \quad t_1 \in A'(a_1, w_1).\]

For otherwise, \( t_1 \in A'[w_1, v_j] \). We first claim \( G \) has an \( A'-B' \) path \( T_0 \) from \( t_0 \in A'(x_1, w_1) \) to \( t'_0 \in B'[b_1, y_1] \), internally disjoint from \( H' \). For otherwise, by (2.5) and \( u_2 = v_j \), \( G \) has a separation \( (G_1, G_2) \), such that \( V(G_1 \cap G_2) = \{x_1, w_1, u_2, u, x_2, y_2\} \) with \( u = y_1 \) (when \( z_1 \neq z_2 \)) or \( u = z_1 \) (when \( z_1 = z_2 \)), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(X_1 \cup X_2) \subseteq V(G_2) \), and \( (G_2, x_1, w_1, u_2, u, x_2, y_2) \) is planar, a contradiction to Lemma 2.0.3.

We may assume \( T_0 \) is disjoint from \( T_1, T_2 \). For otherwise, let \( v \in V(T_0 \cap (T_1 \cup T_2)) \), such that \( T_0[v, t'_0] \) is minimal. Now, if \( v \in T_1 \), then \( T_1[t_1, v] \cup T_0[v, t'_0], T_2 \) contradict the choice of \( T_1, T_2 \); if \( v \in T_2 \), then \( T_1, T_2[t_2, v] \cup T_0[v, t'_0] \) form a cross, contradicting (2.2).
But then, either $T_0, T_2$ contradict the choice of $T_1, T_2$, or $T_0, T_1$ form a cross, contradicting (2.2). □

Now, combined with (2.15) and Lemma 3.0.1, the path from $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t'_1]$ from $a_1$ to $b_1$, the path $A'[a_1, x_1] \cup X_1$ from $a_1$ to $b_2$, the path $A'[t, a_2] \cup T \cup B'[t', y_1] \cup A'_0$ from $a_2$ to $a_0$, and the path $B'_0 \cup B'[t_2, c_2] \cup T_2 \cup A'[w_1, t_2] \cup W[w_1, y_2]$ from $b_1$ to $b_2$ show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

**Lemma 5.0.12** None of (i) and (ii) of Lemma 5.0.8 holds.

**Proof.** For, suppose (i) or (ii) of Lemma 5.0.8 holds.

(i) When (i) of Lemma 5.0.8 holds, by Lemma 5.0.9, we may choose 2-cut $\{z'_1, z'_2\}$ so that $z'_1 \in A'[x_1, w_1]$.

(ii) When (ii) of Lemma 5.0.8 holds, by Lemma 5.0.11, we may assume $N(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2])$. For notation convenience, we let $z'_1 := w_1$ and $z'_2 := z_1$.

(1) $z'_2 \notin V(X_2)$.

For, suppose $z'_2 \in V(X_2)$. Since $z_1 \notin V(X_2)$ by Lemma 5.0.7, we may assume (i) holds. Then $z'_1 = x_1$; or else, it contradicts Lemma 5.0.1 that $H' - y_1 - A'(x_1, x_2)$ contains disjoint paths from $N(y_1) - V(A')$, $y_2$ to $x_1, x_2$, respectively. But now, $\{x_1, y_2, z'_2\}$ is a cut in $G$ separating $V(X_1(x_1, y_2))$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

Since $z'_2 \notin V(X_2)$, then $w_m \in W(z'_2, y_2)$. Now, we let $h \in \{2, \ldots, m\}$, such that $w_h \in W(z'_2, y_2)$, and subject to this, $h$ is minimum.

(2) $H_0 - W[w_h, y_2] \cup A'(x_1, w_1)$ contains a path $Y$ from $N(y_1) - V(A')$ to $x_1$.

Let $v \in N(y_1) \cap V(H_0)$, such that $v \notin V(W[w_h, y_2] \cup A'(x_1, w_1))$. For, suppose such a path $Y$ does not exist. Then, combined with the planar structure of $H_0$, there exist $z''_1 \in A'[x_1, z'_1], z''_2 \in W[w_h, y_2]$, such that $z''_1, z''_2$ are incident with a common finite face of $H$.
and \( \{z''_1, z''_2\} \) is a 2-cut in \( H_0 \). Hence, (i) holds. But then, \( \{z''_1, z''_2\} \) contradicts the choice of \( \{z'_1, z'_2\} \). □

Now, we let \( Y_1 \) be the path obtained from \( Y \) by adding the vertex \( y_1 \) and the edge from \( y_1 \) to the end of \( Y^* \), such that \( Y_1 \) is a path from \( y_1 \) to \( x_1 \), and let \( Y_2 := W[y_2, w_h] \cup w_hu_h \), and so \( Y_2 \) is a path from \( y_2 \) to \( u_h \), disjoint from \( Y_1 \).

(3) \( H_0 - W[z_2, w_m] \) has a path \( Y'_2 \) from \( y_2 \) to \( z'_1 \), internally disjoint from \( A' \).

For otherwise, by the planar structure of \( H_0 \), there exist \( z''_1 \in A'[x_1, z'_1], z''_2 \in W[z_2, w_m] \), such that \( z'_1, z''_2 \) are incident with a common finite face of \( H_0 \), and \( \{z''_1, z''_2\} \) is a 2-cut in \( H_0 \). Hence, (i) holds. But then, \( \{z'_1, z'_2\} \) contradicts the choice of \( \{z'_1, z'_2\} \). □

Now, we let \( Y'_1 := Z'_2 \cup W[z_2, w_m] \cup w_mv_m \), and so \( Y'_1 \) is a path from \( y_1 \) to \( x_2 \), disjoint from \( Y'_2 \).

(4) \( G \) has no cross \( C, D \) from \( c, d \in A' \) to \( c', d' \in B'[b_1, y_1] \), such that \( C, D \) are internally disjoint from \( A', B', H', c \in A'[a_1, z'_1] \), and \( d \in A'(c, x_2) \).

For otherwise, \( C, D, Y'_1, Y'_2 \) force a doublecross, a contradiction. □

(5) If \( u_h \neq x_2 \) and \( G \) has an \( A'-B' \) path \( S \) from \( s \in A'(u_h, x_2) \) to \( s' \in B'[b_1, y_1] \),

internally disjoint from \( H' \), then \( b_1 = r_1 = r' = s' \) and \( S \) is an edge from \( s \) to \( s' \).

We may first assume \( S \) and \( R \) are disjoint; otherwise, \( S, R \) are contained in an \( A'-B' \) bridge, which contradicts (v) of Lemma 4.0.9 due to the path \( u_hw_h \cup W[w_h, y_2] \) from \( u_h \) to \( y_2 \).

Now, \( s' \notin B'(r', y_1) \); otherwise, \( S, R, Y_1, Y_2 \) form a doublecross by \( u_h \neq x_2 \). Thus, \( G \) has no \( A'-B' \) path from \( A'(u_h, x_2) \) to \( B'(r', y_1) \), which further implies that \( S, Q \) are disjoint.

We may assume \( b_1 = r_1 \) and so \( a_1 = x_1 \) by (iii) of Lemma 5.0.4. For, suppose \( b_1 \neq r_1 \).

Then \( s' \neq b_1 \); otherwise, \( s = x_2 = a_2 \), and \( S \) is an edge from \( a_2 \) to \( b_1 \), a contradiction. So \( s' \notin B'(b_1, r') \). But then, \( A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup A'[s, a_2] \cup Y_1 \cup A'_0 \) and

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\[ B'_1 \cup Q \cup A'[q, u_h] \cup Y_2 \cup B'[y_2, b_2] \] show that \( \gamma \) is feasible, a contradiction. \((B'_1, A'_0)\) are defined in (ii) of Lemma 5.0.10.

We may also assume \( r_1 = r' \). For, suppose \( r_1 \neq r' \). By Lemma 5.0.2, there exist a core \( H' \) of \( A, B \) with \( r_1, r_2 \) as feet and \( r' \in B'(r_1, r_2) \), and an \( A'-B' \) bridge \( M_4 \) with extreme hands \( l_4, r_4 \) and feet \( l'_4, r'_4 \), such that \( R \) is internally disjoint from \( M_4, l_4 = r_4 = x_i \) for some \( i \in [2] \), and \( r' \in B'(l'_4, r'_4) \). Since \( G \) has no \( A'-B' \) path from \( A'(u_h, x_2) \) to \( B'(r_1, y_1) \), then \( i \neq 2 \) and \( S \) is internally disjoint from \( M_4 \). Next, \( s' \neq r' \). For, suppose \( s' = r' \). Let \( P' \) be the path from \( l'_4 \) to \( r'_4 \) in \( M_4 \), internally disjoint from \( A', B' \), then \( A'[a_1, r] \cup R \cup S \cup A'(u_h, a_2) \cup Y_1 \cup A'_0 \) and \( B'[b_1, l'_4] \cup P' \cup B'[r'_4, q'] \cup Q \cup A'[q, u_h] \cup Y_2 \) show that \( \gamma \) is feasible, a contradiction. So \( s' \in B'[r_1, r'] \) and \( s = x_2 \) (by the definition of \( r' \)). Now, we see that \( s' \notin B'(r_1, r') \) and \( S \) is not contained in an \( A'-B' \) bridge. For otherwise, by Lemma 4.0.9, \( S \) is contained in \( H' \), which further implies \( x_2 \) is an extreme hand of \( H' \). So \( H' \) is a main core of \( A, B \), a contradiction to Lemma 4.0.8. Therefore, \( s' = r_1 \), and \( S \) is an edge from \( b_1 \) to \( x_2 \), which implies \( a_2 \neq x_2 \), a contradiction.

Thus, now, \( b_1 = r_1 = r' = s' \). To finish (5), we just need to prove that \( S \) is an edge from \( A'(u_h, x_2) \) to \( s' \). For otherwise, \( S \) is contained in an \( A'-B' \) bridge \( N \) with extreme hands \( n_1, n_2 \). Obviously, \( V(N \cap B') \subseteq \{b_1, y_1\} \) (by \( s' = r' \)). Moreover, by Lemma 5.0.3, \( V(N \cap A'(u_h, x_2)) = \emptyset \). Hence, \( n_1 \in A'[x_1, u_h] \) and \( n_2 = x_2 \). By (v) of Lemma 4.0.9, \( H' - y_1 \) does not have a path from \( A'(n_1, n_2) \) to \( y_2 \), internally disjoint from \( A' \). So, by the existence of path \( Y_2, n_1 \notin A'[x_1, u_2] \). So \( n_1 = u_h \). But then, \( \{n_1, n_2, b_1, y_1\} \) is a cut in \( G \) separating \( V(N) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. \( \square \)

(6) \( x_1 \neq z'_1 \).

For, suppose \( x_1 = z'_1 \). Since \( w_1 \neq x_1 \), then (i) holds. And \( G \) has an \( A'-B' \) path from \( A'(u_h, x_2) \) to \( B'[b_1, y_1] \) internally disjoint from \( H' - y_1 \); otherwise, \( G \) has a separation \((G_1, G_2)\) of order 5, such that \( V(G_1 \cap G_2) = \{x_1, z'_2, u_h, x_2, y_2\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1) \), and \( V(X_1 \cup X_2) \subseteq V(G_2) \), a contradiction. Hence, \( A'(u_h, x_2) \neq \emptyset \), and by (5), \( b_1 = r_1 = r' \), and \( a_1 = x_1 \) (by (iii) of Lemma 5.0.4). But then, \( G \) has a separation
(\(G_1, G_2\)) of order 6, such that \(V(G_1 \cap G_2) = \{x_1, z'_2, u_h, x_2, y_2, b_1\}\), \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(X_1 \cup X_2) \subseteq V(G_2)\), and \((G_2, x_1, y_2, x_2, b_1, u_h, z'_2)\) is planar, a contradiction to Lemma 2.0.3. \(\square\)

(7) \(b_2 = y_2\).

For, suppose \(b_2 \neq y_2\). By Lemma 5.0.8, \(N(b_2) = \{y_2, x_1\}\) and \(a_1 \neq x_1\). Now, let \(e' = a_1b' \in E(G)\) with \(b' \in B'(b_1, r_1) \cup B'[y_2, b_2]\). By (i) of Lemma 5.0.4, \(b' \in B'(b_1, r_1)\). Now, since \(x_1 \neq z'_1\), combined with Lemma 3.0.1, the path \(A_0' \cup Y'_1 \cup A'[x_2, a_2]\) from \(a_0\) to \(a_2\), the path \(B'_1 \cup Q \cup A'[z'_1, q] \cup Y'_2 \cup B'[y_2, b_2]\) from \(b_1\) to \(b_2\), the path \(e' \cup B'[b_1, b']\) from \(a_1\) to \(b_1\), and the path \(A'[a_1, x_1] \cup e\) from \(a_1\) to \(b_2\) show that \(\alpha(A', B') = 2\), a contradiction to (i) of Lemma 5.0.10. \(\square\)

(8) \(G\) has an \(A'-B'\) path from \(A'[a_1, z'_1]\) to \(B'(b_1, y_1)\), internally disjoint from \(H'\).

For, suppose \(G\) has no \(A'-B'\) path from \(A'[a_1, z'_1]\) to \(B'(b_1, y_1)\), internally disjoint from \(H'\). Then \(a_1 = x_1\). Now, by (5) and (7), when (i) holds, \(\{b_1, b_2, z'_1, z'_2, u_h\}\) is a cut in \(G\) separating \(a_1, a_2\) from \(a_0\), a contradiction; when (ii) holds, \(\{b_1, b_2, z'_1, y_2, u_h\}\) (when \(z_1 \neq w_2\)) or \(\{b_1, b_2, z'_1, z_1, u_h\}\) (when \(z_1 = w_2\)) is a cut in \(G\) separating \(a_1, a_2\) from \(a_0\), a contradiction. \(\square\)

(9) If \(u_h \neq x_2\), then \(G\) has no \(A'-B'\) path from \(A'[u_h, x_2]\) to \(B'[b_1, y_1]\), internally disjoint from \(H'\).

Suppose \(G\) has an \(A'-B'\) path \(S\) from \(s \in A'[u_h, x_2]\) to \(s' \in B'[b_1, y_1]\), internally disjoint from \(H'\). Then by (5), \(S\) is an edge from \(s\) to \(b_1\) with \(b_1 = r_1 = r' = s'\). So \(s \neq a_2\), and \(s \in A'(u_h, a_2)\).

By (8), \(G\) has an \(A'-B'\) path from \(A'[a_1, z'_1]\) to \(B'(b_1, y_1)\), internally disjoint from \(H'\). Now, this path together with \(S\) forms a cross, which contradicts (4). \(\square\)

(10) \(G\) has disjoint \(A'-B'\) paths from \(A'[a_1, z'_1]\) to \(B'[b_1, y_1]\), internally disjoint from \(H'\).
For otherwise, there exists a vertex \( v \in V(G) \), such that \( G - v \) has no \( A'-B' \) path from \( A'[a_1, z'_1] \) to \( B'[b_1, y_1] \), internally disjoint from \( H' \). Then combined with (10), there exists a separation \((G_1, G_2)\) in \( G \) of order 4, such that \( V(G_1 \cap G_2) = \{v, z'_1, z'_2, u_h\}, a_0, y_1 \in V(G_1 - G_2) \), and \( a_1, a_2, b_2 \in V(G_2) \).

Now, we claim that \((G_2, v, z'_1, z'_2, u_h, a_2, b_2, a_1)\) is planar. Obviously, when \( v \in A' \), this claim is true. So we may assume \( v \notin A' \). Furthermore, if \( v \in B' \) and all the \( A'-B' \) paths from \( v \) to \( A' \) are edges, then our claim is still true. Therefore, we may assume there exists some \( A'-B' \) bridge \( N \) with feet \( n'_1, n'_2 \) and extreme hands \( n_1, n_2 \), such that \( v \in N \). By (v) of Lemma 4.0.9, \( H' - y_1 \) does not contain a path from \( A'(n_1, n_2) \) to \( y_2 \), internally disjoint from \( A' \). Now, \( v \notin B' \); otherwise, \( n'_1 = n'_2 = v \), and \( \{n_1, n_2, v\} \) is a cut in \( G \) separating \( V(N) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. So \( v \notin V(A' \cup B') \). Now, \( N \) has a separation \((N', N'')\) of order 1, such that \( V(N' \cap N'') = \{v\}, n_1, n_2 \in V(N' - N'') \), and \( n'_1, n'_2 \in V(N'' - N') \). We see that \( V(N') = \{n_1, n_2, v\} \); or else, \( \{n_1, n_2, v\} \) is a cut in \( G \) separating \( V(N') - \{n_1, n_2, v\} \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. So, \( V(N') = \{n_1, n_2, v\} \). \( N' \) is planar, and \((G_2, v, z'_1, z'_2, u_h, a_2, b_2, a_1)\) is planar. So our claim is true.

Now, we see that if \( v = a_1, u_h = a_2 \), then \( \{v, z'_1, z'_2, u_h, b_2\} \) is a cut in \( G \) separating \( V(X_1 \cup X_2) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction; if \( v = a_1, u_h = a_2 \) or \( v = a_1, u_h \neq a_2 \), then Lemma 2.0.3 applies; if \( v = a_1, u_h \neq a_2 \), then Lemma 2.0.4 applies. \( \square \)

By (10), we let \( T_1, T_2 \) be two disjoint \( A'-B' \) paths from \( t_1, t_2 \in A'[a_1, z'_1] \) to \( t'_1, t'_2 \in B'[b_1, y_1] \), such that \( T_1, T_2 \) are internally disjoint from \( H', a_1, t_1, t_2, a_2 \) occur on \( A' \) in order, and subject to this, \( A'[t_1, t_2] \) and \( B'[t'_1, t'_2] \) are maximal. By (4), \( T_1, T_2 \) do not form a cross, and so \( b_1, t'_1, t'_2, y_1 \) occur on \( B' \) in order.

(11) \( Q \) is internally disjoint from \( T_1, T_2, t'_1 \in B'[b_1, r'] \), and \( t'_2 \notin B'(q', y_1) \). For, suppose \( Q \) is not internally disjoint from \( T_j \) for some \( j \in [2] \), then \( Q, T_j \) are contained in a same \( A'-B' \) bridge. But then, the existence of the path from \( z'_1 \) to \( y_2 \) in \( H' - y_1 \) contradicts (v) of Lemma 4.0.9.

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Hence, \( t'_2 \notin B'(q', y_1) \) (to void the cross contradicting (4)).

We also see that \( t'_1 \in B'[b_1, r'] \). For otherwise, \( t'_1 \in B'(r', t'_2) \). Now, \( V(R \cap (T_1 \cup T_2)) = \emptyset \). For, suppose there exists \( u \in V(R \cap (T_1 \cup T_2)) \), then we choose \( u \) so that \( R[r', u] \) is minimal. Now, if \( u \in T_1 \), then \( R[r', u] \cup T_1[u, t_1] \) and \( T_2 \) contradict the choice of \( T_1, T_2 \); if \( u \in T_2 \), then \( T_1, R[r', u] \cup T_2[u, t_2] \) form a cross, contradicting (4). So \( V(R \cap (T_1 \cup T_2)) = \emptyset \). But then if \( r \in A'[a_1, t_1] \), \( R \) and \( T_2 \) contradict the choice of \( T_1, T_2 \); if \( r \in A'(t_1, q) \), then \( T_1, R \) form a cross, contradicting (4). □

We let \( Q_0 \) be an \( A'-B' \) path from \( q_0 \in A'(z'_1, a_2) \) to \( q'_0 \in B'[b_1, y_1] \), internally disjoint from \( H' \), such that \( B'[q'_0, y_1] \) is minimal. By the existence of \( Q \), obviously, \( q'_0 \in B'[q', y_1] \).

(12) There do not exist \( c_1, c_2 \in V(G'_0) \), such that \( c_1 \in B'[b_1, t'_1] \), \( c_2 \in B'[q'_0, y_1] \), and \( c_1, c_2 \) are incident with a common finite face of \( G'_0 \).

For, suppose (12) fails. We choose \( c_1, c_2 \) so that \( B'[c_1, c_2] \) is maximal. Since \( t'_1 \in B'[b_1, r'] \), then \( c_1 \in B'[b_1, r'] \). We may further assume \( c_1 \in B'[b_1, r_1] \). In fact, by (iii) of Lemma 5.0.2, when \( r' \neq r_1 \), we have \( r' \in B'(r_1, r_2) \), and so \( r', r_1, r_2 \) are incident with a common finite face of \( G'_0 \), which further implies \( c_1 \in B'[b_1, r_1] \) by the choice of \( c_1, c_2 \).

Now, we may assume \( G \) has an \( A'-B' \) path \( T_3 \) from \( t'_3 \in B'(b_1, c_1) \cup B'(c_2, y_1) \) to \( t_3 \in A' \). For otherwise, \( \{b_1, b_2, c_1, c_2, y_1\} \) is a cut in \( G \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction.

We may assume \( t'_3 \in B'(c_2, y_1) \). For otherwise, \( t'_3 \in B'(b_1, c_1) \), and so \( t'_3 \in B'(b_1, r_1) \), which further implies \( T_3 \) is an edge. Now, by the choice of \( T_1, T_2 \), and by (4) and (9), we have \( t_3 = u_h = a_2 \). Thus, \( A'[a_1, t_1] \cup T_1 \cup B'[t'_3, t'_1] \cup T_3 \cup Y'_1 \cup A'_0 \) and \( B'_1 \cup Q \cup A'[z'_1, q] \cup Y'_2 \) show that \( \gamma \) is feasible, a contradiction.

Now, by the choice of \( Q_0, t_3 \notin A'(z'_1, a_2) \) and \( T_3, Q_0 \) are disjoint. Moreover, by (4), to forbid the cross \( T_3, Q_0, t_3 \notin A'[a_1, z'_1] \), and so \( t_3 = z'_1 \).

We claim that \( G'_0 - B'[t'_1, q'_0] - A'_0 \) contains a path \( B'_3 \) from \( b_1 \) to \( t'_4 \). For otherwise, by maximality of \( B'[c_1, c_2] \), there exists a vertex \( c_3 \in V(A'_0) \), such that \( \{c_2, c_3\} \) is a cut in

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For otherwise, let \( v = z_2' \) when (i) holds; and let \( v = z_2 \) when (ii) holds. Then there exists a 2-cut \( \{ z_1'', z_2'' \} \) in \( H_0 \), such that \( z_1'' \in A'(x_1, z_1') \), \( z_2'' \in W[v, y_2] \), and \( z_1'', z_2'' \) are incident with a common finite face of \( H_0 \). Hence, (i) holds. But then \( \{ z_1'', z_2'' \} \) contradicts the choice of \( \{ z_1', z_2' \} \). \( \square \)

(16) \( G \) has no \( A'-B' \) path from \( A'(t_1, z_1') \) to \( B'(t'_1, q'_0) \), disjoint from \( T_1, Q_0 \).
For, suppose $G$ has an $A'-B'$ path $T$ from $t \in A'(t_1, z'_1]$ to $t' \in B'(t'_1, q'_0)$, disjoint from $T_1, Q_0$. When (i) holds, we let $B^*$ be the path from $b_1$ to $b_2$ in $B'_1 \cup B'(t'_1, q'_0) \cup T \cup A'[t, z'_1] \cup Y'_2$; when (ii) holds, we let $B^*$ be the path from $b_1$ to $b_2$ in $B'_1 \cup B'(t'_1, q'_0) \cup T \cup W'[t, y_2]$. Now, combined with Lemma 3.0.1, the path $B^*$ from $b_1$ to $b_2$, the path $A'[q_0, a_2] \cup Q_0 \cup B'[q'_0, y_1] \cup A'_0$ from $a_2$ to $a_0$, the path $A'[a_1, t_1] \cup T_1 \cup B'[t_1, t'_1]$ from $a_1$ to $b_1$, and the path $A'[a_1, x_1] \cup X_1$ from $a_1$ to $b_2$ show that $\alpha(A', B') = 2$, a contradiction to (i) of Lemma 5.0.10. □

(17) \( t'_2 = q'_0 \).

For otherwise, by (16), $V(T_2 \cap Q_0) \neq \emptyset$. So $T_2, Q_0$ are contained in a same $A'-B'$ bridge. But the existence of the path from $z'_1$ to $y_2$ in $H' - y_1$ contradicts (v) of Lemma 4.0.9. □

(18) $G$ has an $A'-B'$ path $R^*$ from $r'$ to $A'(x_1, z'_1)$, and $t'_1 = r'$.

We may assume $G$ has an $A'-B'$ path from $r'$ to $A'(x_1, z'_1)$. For otherwise, $R$ is disjoint from $T_2$, and $R, T_2$ form a cross, contradicting (4).

Now, we prove $t'_1 = r'$. For otherwise, $r' \in B'(t'_1, q'_0)$. Now, by (16), $V(R^* \cap (T_1 \cup Q_0)) \neq \emptyset$. Obviously, by the definition of $r'$, $V(R^* \cap T_1) = \emptyset$. Thus, $R^*, Q_0$ are contained in a same $A'-B'$ bridge. But then, the path from $z'_1$ to $y_2$ in $H' - y_1$ contradicts (v) of Lemma 4.0.9. □

Now, the path $A'[a_1, x_1] \cup X* \cup A'[z'_1, a_2]$ from $a_1$ to $a_2$ and the path $B'[b_1, r'] \cup R \cup A'[r, t_2] \cup T_2 \cup B'[t'_2, y_1] \cup Z'_2 \cup W[z_2, y_2]$ from $b_1$ to $b_2$ show that $G'_0$ does not contain a path from $B'(t'_1, t'_2)$ to $a_0$, internally disjoint from $B'$; or else, it contradicts (i) of Lemma 3.0.2. So, there exist $c_1 \in B'[b_1, t'_1]$ and $c_2 \in B'[t'_2, y_2]$, such that $c_1, c_2$ are incident with a common finite face of $G'_0$, a contradiction to (12). □
CHAPTER 6
SLIM CONNECTOR

In this chapter, we deal with the case, when no ideal frame in \( \gamma = (G,a_0,a_1,a_2,b_1,b_2) \) admits a fat connector.

**Definition.** Let \( A,B \) be an ideal frame in \( \gamma \) w.r.t. \( a_0 \). Assume there does not exist any fat connectors of ideal frame \( A,B \), then let \( G_0 := G - A \). By Lemma 2.0.6 and the structure of slim connectors, \( G_0 \) has a disk representation with \( B \) and \( a_0 \) occurring on the boundary of the disk, and any \( A-B \) path in \( \gamma \) is induced by a single edge.

**Lemma 6.0.1** Suppose \( \gamma \) is infeasible, and \( A',B' \) is a core \( a_0 \)-frame in \( \gamma \). Let \( a_{-1} := a_2 \) and \( a_3 := a_0 \). Then

(i) There do not exist \( i \in \{0, 1, 2\} \), a graph \( H \) and vertices \( s,s' \in V(H) \), such that \( G \) is obtained from \( H \) by identifying \( s \) with \( s' \), and \( (H,a_{i-1},b_1,a_{i+1},b_2) \) is planar.

(ii) For any \( i \in \{0, 1, 2\} \), \( (G - a_{i-1},a_i,b_1,a_{i+1},b_2) \) or \( (G - a_{i+1},a_i,b_1,a_{i-1},b_2) \) is not planar.

(iii) There do not exist a permutation \( \pi \) of \( \{0, 1, 2\} \), a graph \( H \) and \( s,t,s',t' \in V(H) \), such that \( G \) is obtained from \( H \) by identifying \( s \) with \( s' \) and \( t \) with \( t' \), respectively, \( (H,a_{\pi(0)},b_1,a_{\pi(1)},s,t,s',t',a_{\pi(2)},b_2) \) is planar, and \( a_{\pi(1)},t,s',a_{\pi(2)} \) are distinct in \( H \).

**Proof.** Let \( n \) denote the number of vertices in \( G \). Obviously, \( |E(G)| \geq 3n - 7 \).

For, suppose (i) fails, and there exist \( i \in \{0, 1, 2\} \), a graph \( H \) and vertices \( s,s' \in V(H) \), such that \( G \) is obtained from \( H \) by identifying \( s \) with \( s' \), and \( (H,a_{i-1},b_1,a_{i+1},b_2) \) is planar. Obviously, \( |E(H)| \geq |E(G)| \geq 3n - 7 \). Moreover, we let \( H' := H + \{a_{i-1}b_1,a_{i-1}b_2,a_{i+1}b_1,a_{i+1}b_2,b_1b_2\} \), then \( H' \) is planar with \( |V(H')| = n+1 \) and \( |E(H')| \geq \)
3n − 2 = 3(n + 1) − 5. But now, it contradicts that a planar graph with n + 1 (≥ 3) vertices has at most 3(n + 1) − 6 edges.

For, suppose (ii) fails, and for some i ∈ {0, 1, 2}, (G − a_{i−1}, a_i, b_1, a_{i+1}, b_2) and (G − a_{i+1}, a_i, b_1, a_{i−1}, b_2) are planar. Without loss of generality, we assume i = 0, and the degree of a_1 in G is no more than the degree of a_2 in G. Let k denote the degree of a_1 in G, and let \( G' := G + \{a_2b_1, a_2b_2, a_0b_1, a_0b_2, b_1b_2\} \). Obviously, \( G' - a_1 \) is planar. We also see that a_2 has degree at least \( k + 2 \) in \( G' \), a_0 has degree at least 6 in \( G' \), and b_j has degree at least 5 in \( G' \) for \( j \in [2] \). Moreover, all other vertices of \( G' \) not in \( \{a_0, a_1, a_2, b_1, b_2\} \) have degree at least 6 in \( G' \). Hence, the sum of degrees of each vertices in \( G' - a_1 \) is at least \( 6(n - 5) + (k + 2) + 6 + 5 + 5 = 6n - 12 \). So the number of edges in \( G' - a_1 \) is at least \( 3n - 6 = 3(n - 1) - 3 \). But now, it contradicts that a planar graph with \( n - 1 \) (≥ 3) vertices has at most \( 3(n - 1) - 6 \) edges.

For, suppose (iii) fails due to some permutation \( \pi \) of \{0, 1, 2\}, a graph \( H \) and \( s, t, s', t' \in V(H) \). Obviously, \( |E(H)| \geq |E(G)| \geq 3n - 7 \). Moreover, we let \( H' := H + \{b_1a_{\pi(0)}, b_1a_{\pi(1)}, b_2a_{\pi(0)}, b_2a_{\pi(2)}, a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s'\} \). Since \( G^* \) is 6-connected and \( (H, a_{\pi(0)}, b_1, a_{\pi(1)}, s, t, s', t', a_{\pi(2)}, b_2) \) is planar, then \( a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s' \notin E(H) \), and so \( H' \) is planar with \( |V(H')| = n + 2 \) and \( |E(H')| \geq 3n + 1 = 3(n + 2) - 5 \). But now, it contradicts that a planar graph with \( n + 2 \) (≥ 3) vertices has at most \( 3(n + 2) - 6 \) edges. □

**Definition.** Let \( \gamma = (G, a_0, a_1, a_2, b_1, b_2) \) be a rooted graph with an ideal frame \( A, B \) w.r.t. \( a_0 \). Let \( a'b', a''b'' \in E(G) \) with \( a', a'' \in V(A) \) and \( b', b'' \in V(B) \) all distinct. We say that \( a'b', a''b'' \) form a *cross* (w.r.t. \( A, B \)) if \( a_1, a', a'', a_2 \) occur on \( A \) in order, and \( b_1, b', b'', b_2 \) occur on \( B \) in order. We say that \( a'b', a''b'' \) are *parallel* if \( a_1, a', a'', a_2 \) occur on \( A \) in order, and \( b_1, b', b'', b_2 \) occur on \( B \) in order.

For \( i = 5, 6, 7 \), let \( e_i = a_ib_i \in E(G) \) with \( a_i \in V(A) \) to \( b_i \in V(B) \). We say that \( (e_5, e_6, e_7) \) is a *3-edge configuration* (w.r.t. \( A, B \)) if \( b_6 \in B(b_5, b_7) \) and \( a_1, a_2, a_6 \notin A[a_5, a_7] \).

For \( i = 3, 4, 5, 6, 7 \), let \( e_i = a_ib_i \in E(G) \) with \( a_i \in V(A) \) and \( b_i \in V(B) \). We say that
\((e_3, e_4, e_5, e_6, e_7)\) is a 5-edge configuration (w.r.t. \(A, B\)) if

- \((e_5, e_6, e_7)\) is a 3-edge configuration w.r.t. \(A, B\),
- \(A[a_5, a_7] \subseteq A(a_3, a_4)\), and
- \(b_3, b_4 \in B(b_j, b_5) \cap B(b_j, b_7)\) for some \(j \in [2]\).

**Lemma 6.0.2** Suppose \(\gamma\) is infeasible, and \(A', B'\) is a core \(a_0\)-frame in \(\gamma\). Suppose \(A, B\) is an ideal frame w.r.t. \(a_0\) in \(\gamma\). Then there exists a 5-edge configuration w.r.t. \(A, B\).

**Proof.** (1) For any \(i \in [2]\), \(G\) has a cross from \(A - a_i\) to \(B\).

For, suppose (1) fails. Without loss of generality, we assume \(G\) has no cross from \(A - a_2\) to \(B\). Now, we let \(b' \in B[b_1, b_2]\), such that \(G\) has an edge \(e'\) from \(b'\) to \(A[a_1, a_2]\), and subject to this, \(B[b', b_2]\) is minimal.

We first see that \(G\) has an edge from \(a_2\) to \(B[b_1, b']\); otherwise, \((G, a_1, a_2, b_2, a_0, b_1)\) is planar, a contradiction to (i) of Lemma 6.0.1.

Now, we let \(u_1, u_2 \in B[b_1, b']\), such that \(G\) has an edge from \(u_k\) to \(a_2\) for each \(k \in [2]\), and subject to this, \(B[u_1, u_2]\) is maximal.

We claim that \(G\) has an edge \(e\) from \(b \in B(u_1, u_2)\) to \(a \in A[a_1, a_2]\). For otherwise, we can obtain a new graph \(H\) from \(G\) by splitting \(a_2\) as \(s, s'\), such that \(H\) has no edge from \(B[u_1, u_2]\) to \(s'\) and no edge from \(B[b', b_2]\) to \(s\), and \((H, a_1, b_2, a_0, b_1)\) is planar, which contradicts (i) of Lemma 6.0.1.

We also see that \(a \notin A(a_1, a_2)\). For otherwise, let \(e^* = a_1b^* \in E(G)\) with \(b^* \neq b\). Since \(G\) has no cross from \(A - a_2\) to \(B\), then \(b^* \in B(b_1, b)\). Now, \((e^*, u_1a_2, e, u_2a_2, e')\) is a 5-edge configuration, a contradiction.

So, \(a = a_1\), and all edges from \(B(u_1, u_2)\) to \(A[a_1, a_2]\) are end in \(a_1\). But now, \((G - a_1, a_2, b_2, a_0, b_1)\) and \((G - a_2, a_1, b_2, a_0, b_1)\) are planar, contradicting (ii) of Lemma 6.0.1.

\(\square\)

We let \(b_1', b_2' \in B[b_1, b_2]\), such that \(b_1, b_1', b_2, b_2\) occur on \(B\) in order, \(G\) has an edge from \(b_i'\) to \(A\) for each \(i \in [2]\), and subject to this, \(B[b_1', b_2']\) is maximal.
(2) For some \(i \in [2]\), \(G\) has no edge from \(b'_i\) to \(A(a_1, a_2)\).

For, suppose \(G\) has an edge \(e'_i\) from \(b'_i\) to \(A(a_1, a_2)\) for each \(i \in [2]\). For each \(k \in [2]\), since the degree of \(a_k\) in \(G\) is at least 4, then we may assume \(G\) has an edge \(e_k\) from \(a_k\) to \(B(b'_1, b'_2)\). But now, \(e_1, e_2, e'_1, e'_2\) form a doublecross, a contradiction. □

By symmetry, without loss of generality, we may assume that \(G\) has no edge from \(b'_1\) to \(A(a_1, a_2)\), and has an edge \(e_3\) from \(b'_1\) to \(a_1\).

By (1), there exist \(e_4 = a_4b_4, e_5 = a_5b_5 \in E(G)\) with \(a_4, a_5 \in A(a_1, a_2)\) and \(b_4, b_5 \in B[b'_1, b'_2]\), such that \(e_4, e_5\) form a cross, and \(b_1, b_4, b_5, b_2\) occur on \(B\) in order. We further choose \(e_4, e_5\) so that \(B[b'_1, b'_4] \cup A[a_1, a_5]\) is minimal, and subject to this, \(B[b_5, b_2] \cup A[a_4, a_2]\) is minimal. By the choice of \(e_4, e_5\), we may assume

(3) \(G\) has no edge from \(B[b_1, b_4]\) to \(A(a_5, a_2)\), no edge from \(A(a_1, a_5)\) to \(B(b_4, b_2)\), no edge from \(b_4\) to \(A(a_4, a_2)\), and no edge from \(a_5\) to \(B(b_5, b_2)\).

(4) \(G\) has no cross from \(B[b_1, b_4]\) to \(A[a_1, a_5]\) and no cross from \(B[b_5, b_2]\) to \(A[a_4, a_2]\).

For otherwise, such a cross together with \(e_4, e_5\) forms a doublecross. □

(5) If \(G\) has an edge from \(B[b_5, b_2]\) to \(A(a_1, a_4)\), then \(G\) has no edge from \(B(b_4, b_5)\) to \(A(a_1, a_4) - a_5\).

For, suppose \(G\) has an edge \(e\) from \(b \in B(b_5, b_2)\) to \(a \in A(a_1, a_4)\) and an edge \(e'\) from \(b' \in B(b_4, b_5)\) to \(a' \in A(a_1, a_4) - a_5\). Now, by (3), we have \(a \notin A(a_1, a_5), a' \notin A(a_1, a_5),\) and so \(a, a' \in A(a_5, a_4)\). But then, \((e_3, e_4, e', e_5, e)\) is a 5-edge configuration. □

Let \(b'_5 \in B(b_4, b_5)\), such that \(G\) has an edge \(e'_5\) from \(a_5\) to \(b'_5\), and subject to this, \(B[b'_5, b_2]\) is maximal.

(6) \(G\) has no edge from \(B(b'_5, b_5)\) to \(A - a_5\).

For, suppose \(G\) has an edge \(e\) from \(B(b'_5, b_5)\) to \(A - a_5\). Then \(b_5 \neq b'_5\), and \((e_3, e_4, e'_5, e, e_5)\) forms a 5-edge configuration, a contradiction. □
(7) $G - a_4 b'_5$ has no cross from $B[b'_5, b_2]$ to $A(a_5, a_2)$.

For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A(a_5, a_2)$ and $b', b'' \in B[b'_5, b_2]$, such that $e', e''$ form a cross, $a_1, a', a'', a_2$ occur on $A$ in order, and $e'' \neq a_4 b'_5$.

Then we may assume $a'' \notin A(a_4, a_2)$; otherwise, $e_5', e', e''$ form a doublecross. So $a' \in A(a_5, a_4)$. Now, we see that $b'' = b'_5$; otherwise, $(e_3, e_4, e'_5, e', e'')$ is a 5-edge configuration. Since $e'' \neq a_4 b'_5$, then $a'' \notin a_4$. So $a'' \in A(a_5, a_4)$. Now, let $e^* = a''b^* \in E(G)$ with $b^* \in B[b_1, b_2]$. Since the degree of $a''$ in $G$ is at least 6, we may further let $b^* \notin \{b', b'', b_4\}$.

First, we see that $b^* \notin B[b_1, b_4]$; otherwise, $e^*, e', e_4, e'_5$ form a doublecross. Next, $b^* \notin B(b_4, b'_5)$; otherwise, $(e_3, e_4, e^*, e'_5, e')$ is a 5-edge configuration. Moreover, $b^* \notin B(b'_5, b')$; otherwise, $(e_3, e_4, e'_5, e^*, e')$ is a 5-edge configuration. So we may assume $b^* \in B(b', b_2)$, but then $(e_3, e_4, e''', e^*, e')$ is a 5-edge configuration, a contradiction. $\Box$

If $a_4 \neq a_2$, we let $b_1^*, b_2^* \in B(b_4, b_2)$, such that $b_1^*, b_2^* \in B$ in order, $G$ has an edge $e^*_i$ from $a_i^* \in A(a_4, a_2)$ to $b_i^*$ for $i \in [2]$, and subject to this, $B[b^*_1, b^*_2]$ is maximal.

(8) If $a_4 \neq a_2$, then $G$ has no edge from $B(b_1^*, b_2^*)$ to $a_5$.

For, suppose $a_4 \neq a_2$, and $G$ has an edge $e^*_5$ from $b^*_5 \in B(b_1^*, b_2^*)$ to $a_5$. We see that $b^*_2 \neq b_2$.

For otherwise, $b^*_2 = b_2$ and $a^*_2 \neq a_2$. By (3), $G$ has no edge from $a_2$ to $B[b_1, b_4]$, and so $G$ has an edge from $a_2$ to $B(b_4, b_2)$, which together with $e_4, e^*_2, e^*_5$ forms a doublecross. Then we shall show that we can obtain a new graph $H$ from $G$ by splitting $a_5$ or $b^*_5$ as $s, s'$, such that $(H, a_1, b_2, a_0, b_1)$ is planar.

We first claim that $G$ has no edge from $B(b_1, b_1^*)$ to $A(a_4, a_2)$ and no cross from $B[b_1, b_1^*]$ to $A[a_1, a_2]$. In fact, we see that $G$ has no edge from $B(b_4, b_1^*)$ to $A[a_1, a_2] - a_4$. For otherwise, let $e = ab \in E(G)$ with $b \in B(b_4, b_1^*)$ and $a \in A[a_1, a_2] - a_4$. Then by the definition of $b_1^*, b_2^*$, we have $a \notin A(a_1, a_2)$. Moreover, $a \neq a_1$ to avoid the doublecross $e, e_4, e_5^*, e_1^*$. But then $a \in A(a_1, a_4)$, and so $(e_3, e_4, e, e_1^*, e_5^*)$ is a 5-edge configuration, a contradiction. Now, combined with (3) and (4), we may assume $G$ has no edge from $B[b_1, b_1^*]$ to $A(a_4, a_2)$ and no cross from $B[b_1, b_1^*]$ to $A[a_1, a_2]$.
We also claim that $G$ has no edge from $B(b_2^*, b_2)$ to $A[a_1, a_2]$. For, suppose $G$ has an edge $e$ from $b \in B(b_2^*, b_2)$ to $a \in A[a_1, a_2]$. Then $a \neq a_1$; otherwise, $(e, e_5^*, e_3^*, e_4, e)$ is a 5-edge configuration. And $a \notin A(a_1, a_3)$; otherwise, $(e_3, e_4, e_5^*, e_2, e)$ is a 5-edge configuration. We may also assume $a \neq a_4$; otherwise, $e_4, e_1^*, e_5^*, e$ form a doublecross. So $a \in A(a_4, a_2)$, but it contradicts the definition of $b_1^*, b_2^*$.

Moreover, we claim that $G - \{a_5, b_5^*\}$ has no edge $e$ from $B(b_1^*, b_2^*)$ to $A[a_1, a_4]$, such that $e \neq a_4b_1^*$. For, suppose there exists $e = ab \in E(G)$ with $e \neq a_4b_1^*$, $a \in A[a_1, a_4] - a_5$, and $b \in B(b_1^*, b_2^*) - b_5^*$. Then we may assume $b \notin B(b_5^*, b_5^*)$; otherwise, if $a \in A[a_1, a_5]$, then $a = a_1$ by (3), and $(e_3^*, e, e_5^*, e_1^*, e_4)$ is a 5-edge configuration; if $a \in A(a_5, a_4)$, then $e_4, e_5^*, e, e_1^*$ form a doublecross. So $b \in B(b_1^*, b_5^*)$. Now, if $a \in A[a_1, a_5]$, then $e_4, e_5^*, e, e_1^*$ form a doublecross. So $a \in A(a_5, a_4)$. We may further assume $b = b_1^*$; or else, $b \in B(b_1^*, b_5^*)$, and $(e_3^*, e_5^*, e, e_1^*, e_4)$ is a 5-edge configuration. Since $e \neq a_4b_1^*$, then $a \in A(a_5, a_4)$. Now, we let $e_0 = ab_0 \in E(G)$ with $b_0 \in B(b_1^*, b_2^*)$. Since the degree of $a$ in $G$ is at least 6, we may further let $b_0 \notin \{b_4, b_1^*, b_5^*\}$. By (3), $b_0 \notin B(b_1^*, b_4)$. Moreover, $b_0 \notin B(b_4, b_1^*)$; or else, $(e_3^*, e_4, e_0, e_1^*, e_4^*)$ is a 5-edge configuration. By $b = b_1^*$, we have $b_0 \notin B(b_1^*, b_2^*) - b_5^*$. So $b_0 \in B(b_2^*, b_2^*)$, but it contradicts that $G$ has no edge from $B(b_2^*, b_2)$ to $A[a_1, a_2]$.

Finally, we claim that $G$ has no cross from $A(a_4, a_2)$ to $B(b_1^*, b_5^*) \cup B(b_5^*, b_2^*)$. For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A(a_4, a_2)$ and $b', b'' \in B(b_1^*, b_5^*) \cup B(b_5^*, b_2^*)$, such that $e', e''$ form a cross, and $a_1, a', a'', a_2$ occur on $A$ in order. Then $b' \in B(b_1^*, b_5^*)$ to avoid the doublecross $e_4, e_5^*, e_1^*, e_4'$. So $b'' \in B(b_5^*, b_2^*)$. Moreover, $a_2^* \in A[a''_2, a_2]$ to avoid the doublecross $e_4, e_5^*, e''_2$. But now, $(e_3^*, e_5^*, e', e'', e_4)$ is a 5-edge configuration.

Now, we let $e' = a'b', e'' = a''b'' \in E(G)$ with $b' \in B(b_1^*, b_5^*)$, $b'' \in B(b_5^*, b_2^*)$, and $a', a'' \in A(a_4, a_2)$, such that $B[b', b'']$ is minimal.

We may assume $G$ has an edge $e_0$ from $b_5^*$ to $a_0 \in A[a_1, a_1] \cup A(a''_2, a_2]$ with $a_0 \neq a_5$. For otherwise, combined with (6) and our claims, we can obtain a new graph $H$ from $G$ by splitting $a_5$ as $s, s'$, such that $(H, a_1, b_2, a_0, b_1)$ is planar, which contradicts (i) of
Lemma 6.0.1.

To avoid the doublecross \( e_5^*, e_0, e'', e_4 \), we may further assume \( a_0 \in A[a_1, a'] \).

Now, we claim that \( G \) has no edge from \( a_5 \) to \( B(b_4, b_2) - b_5^* \). For, suppose \( G \) has an edge \( e \) from \( a_5 \) to \( b \in B(b_4, b_2) - b_5^* \). Assume \( b \in B(b_4, b_5^*) \). Now, if \( a_0 \in A(a_5, a') \), then \( e, e_0, e_4, e' \) form a doublecross. So \( a_0 \in A(a_1, a_5) \). Now, let \( e_6 = a_5b_6 \in E(G) \). Since the degree of \( a_5 \) in \( G \) is at least 6, then we may let \( b_6 \notin \{b_4, b', b_5^*\} \). To avoid the doublecross \( e_6, e_0, e_4, e', b_6 \notin B(b_5^*, b_2) \). Moreover, \( b_6 \notin B(b', b_5^*) \); or else, \( (e_5^*, e_0, e_6, e', e_4) \) is a 5-edge configuration. By (6), \( b_6 \notin B(b_4, b') \). So \( b_6 \in B(b_1, b_4) \). But then \( (e_2^*, e_0, e_4, e', e_4) \) is a 5-edge configuration. So we may assume \( b \in B(b_5^*, b_2) \). By (6), \( b \notin B(b_2', b_2) \). Now, if \( a_0 \in A[a_1, a_5] \), then \( e_0, e, e_4, e' \) form a doublecross; if \( a_0 \in A(a_5, a') \), then \( (e_2^*, e, e_0, e', e_4) \) is a 5-edge configuration, a contradiction.

Hence, by our claims, we can obtain a new graph \( H \) from \( G \) by splitting \( b_5^* \) as \( s, s' \), such that \( (H, a_1, b_2, a_0, b_1) \) is planar, which still contradicts (i) of Lemma 6.0.1. □

For each \( a_j \), we let \( u_1^j, u_2^j \in B[b_1, b_2] \), such that \( b_1, u_1^j, u_2^j, b_2 \) occur on \( B \) in order, \( G \) has an edge \( f_i^j \) from \( a_j \) to \( u_i^j \) for \( i \in [2] \), and subject to this, \( B[u_1^j, u_2^j] \) is maximal.

(9) If \( a_4 \neq a_2 \), then \( G \) has an edge from \( a_2 \) to \( B(b_5, b_2) \).

For, suppose \( a_4 \neq a_2 \) and \( G \) has no edge from \( a_2 \) to \( B(b_5, b_2) \). Since the degree of \( a_2 \) in \( G \) is at least 4, then, combined with \( a_4 \neq a_2 \) and the choice of \( e_4, e_5 \), we have \( u_1^2, u_2^2 \in B(b_1, b_2) \), \( u_1^2 \neq u_2^2 \), and \( G \) has an edge \( f_2 \) from \( a_2 \) to \( B(u_1^2, u_2^2) \). Then we shall show that \( (G, a_1, b_2, a_0, b_1) \) is planar.

We claim that \( G \) has no edge from \( B(u_1^2, u_2^2) \) to \( A[a_1, a_2] \). For, suppose \( G \) has an edge \( e \) from \( b \in B(u_1^2, u_2^2) \) to \( a \in A[a_1, a_2] \). First, \( a \notin A[a_1, a_5] \); otherwise, \( e, e_4, e_5, f_1^2 \) form a doublecross. By (8), \( a \neq a_5 \). So \( a \in A(a_5, a_2) \). Now, we may assume \( b_5 = b_2 \); otherwise, \( (e_5, f_2^2, e, f_1^2, e_4) \) is a 5-edge configuration, a contradiction. Since \( b_5 = b_2 \), then \( u_2^2 \neq b_5 \). But now, \( (e_3, f_1^2, e, f_2^2, e_5) \) is a 5-edge configuration, a contradiction.

We also claim that \( G \) has no cross from \( A[a_1, a_2] \) to \( B[b_1, u_1^2] \). For, suppose there exist \( e' = a'b', e'' = a''b'' \in E(G) \) with \( a', a'' \in A[a_1, a_2] \) and \( b', b'' \in B[b_1, u_1^2] \), such that
$e', e''$ form a cross, and $a_1, a', a'', a_2$ occur on $A$ in order. We see that $b'' \in B(b_4, v^2)$. For otherwise, $b'' \in B(b_1, b_4)$, and by the choice of $e_4, e_5$, $a'' \in A[a_1, a_5]$ and $a' = a_1$. But now $e', e'', e_4, e_5$ form a doublecross, a contradiction. Moreover, $a' = a_1$; otherwise, $(e_3, e'', e', f_2, e_5)$ is a 5-edge configuration. But then, $e', f_2, e_4, e_5$ form a doublecross.

Finally, we claim that $G$ has no parallel edges from $A[a_1, a_2]$ to $B(u^2_2, b_2)$. For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A[a_1, a_2]$ and $b', b'' \in B(u^2_2, b_2)$, such that $e', e''$ are parallel, and $a_1, a', a'', a_2$ occur on $A$ in order. We see that $a' \in A[a_4, a_2]$; otherwise, $e', e'', e_4, f^2_1$ form a doublecross. Moreover, $b'' \in B(u^2_2, b_2)$; otherwise, $e_5, e'', e_4, f^2_1$ form a doublecross. We may assume $b_5 = b_2$; otherwise, $(e_3, e'', e', f^2_1, e_4)$ is a 5-edge configuration. So $u^2_2 \neq b_5$. Now, let $e = e''b \in E(G)$ with $b \notin \{b', b_3\}$. Then $b \notin B(b_1, u^2_1)$ to avoid the doublecross $e, e'', f^2_2, e'$. Moreover, $b \notin B(u^2_2, b')$; otherwise, $(e_3, f^2_1, e, e', e'')$ is a 5-edge configuration. Since $G$ has no edge from $B(u^2_2, u^2_2)$ to $A[a_1, a_2]$, then $b \in B(b', b_3)$. But now, $(e_3, f^2_1, e', e, e_5)$ is a 5-edge configuration.

Hence, by our claims, $(G, a_1, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 6.0.1. \[\square\]

(10) $G$ has no edge from $B(b_5, b_2)$ to $A(a_1, a_4)$.

For, suppose $G$ has an edge $e$ from $b \in B(b_5, b_2)$ to $a \in A(a_1, a_4)$. We choose $e$ so that $B(b, b_2)$ is minimal. By (3), $a \in A(a_5, a_4)$. By (5), $G$ has no edge from $B(b_4, b_3)$ to $A(a_1, a_4) - a_5$. Moreover, since the degree of $a$ in $G$ is at least 6, then we let $e_0 = ab_0$ with $b_0 \in B(b_1, b_2)$ and $b_0 \notin \{b_4, b_5, b\}$. Now, by (3) and (5), and by the definition of $b$, we have $b_0 \in B(b_5, b)$.

$G$ has no edge from $A(a_4, a_2)$ to $B(b_1, b)$. For, suppose there exists $e' = a'b' \in E(G)$ with $a' \in A(a_4, a_2)$ and $b' \in B(b_1, b)$. Then by (3), $b' \notin B(b_1, b_4)$. So $b' \in B(b_4, b)$. But then, $e, e', e_4, e_5$ form a doublecross.

$G$ has no edge from $b_4$ to $A(a_5, a_4)$ or no edge from $a_4$ to $B(b_4, b)$; otherwise, such two edges together with $e_5, e$ form a doublecross, a contradiction.

Now, we see that $G$ has an edge $e'$ from $a_1$ to $b' \in B(b_4, b_2)$; otherwise, since $G$ has no edge from $b_4$ to $A(a_5, a_4)$ or no edge from $a_4$ to $B(b_4, b)$, then combined with (3), (4),
(6), and (7), we can obtain a new graph $H$ from $G$ by splitting $a_4$ or $b_4$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

We also see that $G$ has no edge from $a_1$ to $B(b'_5, b)$; otherwise, such an edge together with $e_3, e_4, e'_5, e$ forms a 5-edge configuration, a contradiction.

Hence, $b' \in B(b_4, b'_5] \cup B[b, b_2]$. We further choose $e'$ so that $B[b', b_2]$ is maximal. Moreover, we let $e'' = a_1b'' \in E(G)$ with $b'' \in B(b_4, b'_5] \cup B[b, b_2]$ so that $B[b'', b_2]$ is minimal.

Now, assume $b'' \in B(b_4, b'_5]$. Then by the choice of $e''$, $G$ has no edge from $a_1$ to $B[b, b_2]$. Moreover, $G$ has no edge from $B[b_1, b_4]$ to $A(a_1, a_2)$; otherwise, by (3), such an edge must end in $A(a_1, a_5)$, which together with $e', e_4, e_5$ forms a doublecross. Hence, $G$ has an edge $e_6$ from $a_4$ to $b_6 \in B(b_4, b_5)$; or else, we can obtain a new graph $H$ from $G$ by splitting $b_4$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1. Now, $G$ has no edge from $b_4$ to $A(a_1, a_4)$; or else, such an edge together with $e_5, e', e_6$ forms a doublecross. So we may assume $a_2 \neq a_4$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1. Then $u_1^{g_2} \in B[b, b_2]$ (by (7) and (9)). Moreover, $b_6 \notin B(b', b_5]$; otherwise, $(f_2^1, e, e_6, e', e_4)$ is a 5-edge configuration. So $G$ has no edge from $a_4$ to $B(b', b_5]$. Therefore, we can obtain a new graph $H$ from $G$ by splitting $a_4$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

So we may assume $b'' \in B[b, b_2]$. Now, $a_2 = a_4$; otherwise, $u_1^{g_2} \in B[b, b_2]$ (by (7) and (9)) and $(f_2^1, e'', e_0, e_5, e_4)$ is a 5-edge configuration.

We also claim that $G$ has an edge $e_6$ from $a_6 \in A(a_1, a_2)$ to $b_6 \in B[b_1, b_4]$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1.

Then $b_6 \notin B[b_1, b_4]$; otherwise, $a_6 \in A(a_1, a_5)$, and $(e, e'', e_5, e_4, e_6)$ is a 5-edge configuration. Hence, $b_6 = b_4$, and $G$ has no edge from $a_5$ to $B[b_1, b_4]$, which further implies $b'_5 \neq b_5$ (as the degree of $a_5$ in $G$ is at least 6).
Now, we may assume $u_2^2 \notin B[b, b_2]$. For, suppose not. Then $G$ has no edge from $\{a_1, a_2\}$ to $B(b_4, b_5)$; otherwise, such an edge together with $f_2^2, e'', e_5, e_6$ forms a 5-edge configuration. Moreover, $a_6 \notin A(a_5, a_2)$; otherwise, $(f_2^2, e'', e_0, e_5, e_6)$ is a 5-edge configuration. But now, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1.

Since $u_2^2 \notin B[b, b_2]$, then $G$ has no edge from $a_2$ to $B[b, b_2]$. By (7), $G$ has no edge from $a_2$ to $B(b_5, b)$. By (3), $G$ has no edge from $a_2$ to $B[b_1, b_4]$. Since the degree of $a_2$ in $G$ is at least 4, then $G$ has an edge $e'_2$ from $a_2$ to $B(b_4, b_5)$. Now, $a_6 \notin A(a_5, a_2)$; otherwise, $e_6, e_5, e, e'_2$ form a doublecross. Moreover, $b' \notin B(b_4, b)$ to avoid the doublecross $e', e'_2, e_6, e$. Hence, combined with (6), we can obtain a new graph $H$ from $G$ by splitting $a_2$ as $s, s'$, such that $(H, a_1, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1. 

Now, by (3), (8), (9), and (10), we have

(11) $G$ has no edge from $A(a_1, a_5) \cup A(a_4, a_2)$ to $B(b_4, b_5)$ and no edge from $B[b_1, b_4] \cup B(b_5, b_2)$ to $A(a_5, a_4)$.

(12) There do not exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A[a_5, a_4]$ and $b', b'' \in B[b_4, b_5]$, such that $e', e''$ are parallel, $a_1, a', a'', a_2$ occur on $A$ in order, and $e' \neq a_5b_4, e'' \neq a_4b_5$.

For, suppose such $e', e''$ exist. Then $b' = b_4$ or $b'' = b_5$; otherwise, $(e_3, e_4, e', e'', e_5)$ is a 5-edge configuration, a contradiction.

We may further assume $b' = b_4$. For otherwise, $b'' = b_5$ and $a'' \neq a_4$. Now, let $e = a''b \in E(G)$ with $b \in B[b_1, b_2]$. Since the degree of $a''$ in $G$ is at least 6, we may further let $b \notin \{b_4, b', b_5\}$. By (11), $b \notin B[b_1, b_4] \cup B(b_5, b_2]$. Moreover, $b \notin B(b_4, b')$; otherwise, $(e_3, e_4, e, e', e'')$ is a 5-edge configuration. So $b \in B(b', b_5)$. But then $(e_3, e_4, e', e, e_5)$ is a 5-edge configuration.

By $b' = b_4$, we have that $G - a_4b_5$ has no parallel edges from $B(b_4, b_5]$ to $A[a_5, a_4]$. 

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Now, since $e'' \neq a_4b_5$ and the degree of $a''$ in $G$ is at least 6, then combined with (11), we may choose $b'' \neq b_5$, and so $b'' \in B(b_4, b_5)$.

Since $e' \neq a_5b_4$, then $a' \in A(a_5, a_4)$. Moreover, since the degree of $a'$ in $G$ is at least 6, we let $e = a'b \in E(G)$ with $b \in B[b_1, b_2]$ and $b \notin \{b_4, b'', b_5\}$. By (11), $b \notin B[b_1, b_4] \cup B(b_5, b_2]$. And $b \notin B(b_4, b'')$; otherwise, $(e_3, e_4, e, e'', e_5)$ is a 5-edge configuration. So $b \in B(b'', b_5)$.

Now, $G$ has no edge from $a_2$ to $B[b_5, b_2]$; otherwise, $(f_2^2, e_5, e, e'', e')$ is a 5-edge configuration. Hence, $a_4 = a_2$ (by (9)), and so $G$ has no edge from $a_4$ to $B[b_5, b_2]$. Moreover, $G$ has no edge from $a_1$ to $B(b_4, b_5)$; otherwise, such an edge together with $e', e_5, e''$ forms a doublecross.

Hence, since $G - a_4b_5$ has no parallel edges from $B(b_4, b_5)$ to $A[a_5, a_4]$, then, combined with (3), (4) and (11), any two edges from $B[b_1, b_4]$ to $A$ do not form a cross, and any two edges from $B(b_4, b_2]$ to $A$ are not parallel, which further implies that $(G, a_1, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

(13) $G$ has an edge $e_0$ from $a_1$ to $b_0 \in B(b_4, b_2)$.

For, suppose $G$ has no edge from $a_1$ to $B(b_4, b_2)$. Then by (3), (4), (11), and (12), we can obtain a new graph $H$ from $G$ by splitting $a_5, a_4$ as $s, s'$ and $t, t'$, respectively, such that $(H, a_0, b_1, a_1, s, t, s', t', a_2, b_2)$ is planar, a contradiction to (iii) of Lemma 6.0.1.

We choose $e_0$ so that $B[b_0, b_2]$ is maximal. Moreover, we let $e_0' = a_1b_0' \in E(G)$ with $b_0' \in B(b_4, b_2]$ so that $B[b_0', b_2]$ is minimal.

(14) $a_4 \neq a_2$.

For, suppose $a_4 = a_2$. We first claim that $A(a_5, a_2) \neq \emptyset$.

For otherwise, $A(a_5, a_2) = \emptyset$. Now, we may assume $G$ has an edge $e$ from $b \in B[b_1, b_4]$ to $a \in A(a_1, a_5)$; or else, combined with (3), (4) and (6), $(G - a_1, a_2, b_2, a_0, b_1)$ is planar and $(G - a_2, a_1, b_2, a_0, b_1)$ is planar, a contradiction to (ii) of Lemma 6.0.1.
Moreover, $G$ has no edge from $a_2$ to $B(b_4, b_5)$. For, suppose $G$ has an edge $e'$ from $a_2$ to $b' \in B(b_4, b_5)$. Then $G$ has no edge from $a_1$ to $B(b_4, b_5)$; or else, such an edge together with $e, e', e_5$ forms a doublecross. So $b_0 \in B[b_5, b_2]$. Now, $G$ has no edge from $a_2$ to $B(b_5, b_2)$; otherwise, such an edge together with $e_0, e'_5, e', e$ forms a 5-edge configuration. So, combined with (3), (4) and (6), $(G, a_1, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Now, since $G$ has no edge from $a_2$ to $B(b_4, b_5)$, then $G$ has an edge from $a_2$ to $B(b_5, b_2)$ (by the degree of $a_2$ in $G$), and so $a_2' \in B(b_5, b_2)$.

Assume $b_0 \in B(b_4, b_5)$. Then $b \notin B[b_1, b_4]$ to avoid the doublecross $e_0, e, e_4, e_5$. Now, by the degree of $a_5$ in $G$, $b'_0 \neq b_5$, and $G$ has an edge $e'_0$ from $a_5$ to $b'_0 \in B(b'_5, b_5)$. By (6), $b_0 \in B(b_4, b'_5)$. So $G$ has no edge from $a_1$ to $B[b_5, b_2]$; otherwise, such an edge together with $f'_2, e'_5, e_0, e$ forms a 5-edge configuration. Hence, combined with (3), (4) and (6), we can obtain a new graph $H$ from $G$ by splitting $b_4$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Therefore, we may assume $G$ has no edge from $a_1$ to $B(b_4, b_5)$. Moreover, $G$ has an edge from $B[b_1, b_4]$ to $A(a_1, a_5)$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1. Hence, we may choose $e$ so that $b \in B[b_1, b_4]$. Then $b'_0 \notin B(b_5, b_2)$ and $b'_0 = b_5$; otherwise, $(f'_2, e'_0, e'_5, e_4, e)$ is a 5-edge configuration. Hence, $b_0 = b'_0 = b_5$, and we can obtain a new graph $H$ from $G$ by splitting $b_5$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Thus, our claim that $A(a_5, a_2) \neq \emptyset$ holds, and there exists $a_6 \in A(a_5, a_2)$. Since the degree of $a_6$ in $G$ is at least 6 and $G$ has no edge from $a_6$ to $B[b_1, b_4] \cup B(b_5, b_2)$ (by (11)), we may let $b'_6, b''_6 \in B(b_4, b_5)$ with $b'_6 \neq b''_6$, such that $b_1, b'_6, b''_6, b_2$ occur on $B$ in order, $G$ has an edge $e'_6$ from $a_6$ to $b'_6$ and an edge $e''_6$ from $a_6$ to $b''_6$, and subject to this, $B[b'_6, b''_6]$ is maximal.

We now claim that $G$ has no edge from $B[b_1, b_4]$ to $A(a_1, a_5)$. For, suppose $G$ has an edge $e''$ from $b''_6 \in B[b_1, b_4]$ to $a'' \in A(a_1, a_5)$. Then $b_0 \in B(b_4, b'_6)$ to avoid the
doublecross $e_0, e''', e_5, e''_6$. Moreover, $b_0 \notin B(b'_6, b_5)$; otherwise, $(e_3, e_4, e''_6, e_0, e_5)$ is a 5-edge configuration. Hence, $b_0 \in B[b_5, b_2]$ and $G$ has no edge from $a_1$ to $B(b_4, b_5)$. We also see that $G$ has no edge from $a_1$ to $B(b_5, b_2)$ or no edge from $a_2$ to $B(b_5, b_2)$; otherwise, such two edges together with $e_5, e''_6, e''$ form a 5-edge configuration. But then, combined with (3), (4), (11), and (12), we can obtain a new graph $H$ by splitting $a_2$ as $s, s'$, such that $(H, a_1, b_0, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Thus, $G$ has no edge from $B[b_1, b_4]$ to $A(a_1, a_5)$, and so by (11) and (12), $(G-a_1, a_2, b_2, a_0, b_1)$ is planar. Now, by (ii) of Lemma 6.0.1, $(G-a_2, a_1, b_2, a_0, b_1)$ is not planar, and so we may assume $G$ has an edge $e$ from $a_1$ to $b \in B(b_4, b_5)$ and an edge $e'$ from $b' \in B[b_1, b]$ to $a' \in A(a_1, a_2)$. And $b \notin B(b'_6, b_5)$ to avoid 5-edge configuration $(e_3, e_4, e''_6, e, e_5)$. Moreover, we may assume $G$ has no edge from $a_2$ to $B(b_4, b_5)$; otherwise, such an edge together with $e, e', e_5$ forms a doublecross. So, by the degree of $a_2$, $u_2 \in B[b_5, b_2]$. But now, $(f_2^2, e_5, e''_6, e, e')$ is a 5-edge configuration. □

Now, by (9) and (14), $G$ has an edge from $a_2$ to $B(b_5, b_2)$, and so $u_2 \in B(b_5, b_2)$. By (3), (11) and (14), $G$ has no edge from $a_2$ to $B[b_1, b_5]$, and so $u_2 \in B[b_5, b_2]$.

(15) $b_0 \in B(b_4, b_5)$.

For otherwise, $b_0 \in B[b_5, b_2]$. Now, we see that $b'_0 \neq b_5$; otherwise, $b_0 = b'_6 = b_5$, and by (3), (4), (11), (12), and (14), we can obtain a new graph $H$ from $G$ by splitting $a_4$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

Then $G$ has no edge from $B[b_1, b_4]$ to $A(a_1, a_5)$; or else, such an edge together with $f_2^2, e''_0, e_5, e_4$ forms a 5-edge configuration. Hence, $A(a_1, a_5) = \emptyset$, and by the degree of $a_5$ in $G$, $b'_5 \neq b_5$. Now, $G$ has no edge from $B[b_5, b'_0]$ to $A[a_4, a_2]$; otherwise, such an edge together with $f_2^2, e''_0, e_5, e_4$ forms a 5-edge configuration.

Moreover, if $b_4a_5 \in E(G)$, then $G$ has no edge from $B(b_4, b_5)$ to $A(a_5, a_2)$; otherwise, such an edge together with $f_2^2, e''_0, e_5, a_5b_4$ forms a 5-edge configuration.

We may assume $u_2 \in B[b_5, b'_0]$. For otherwise, $G$ has no edge from $B[b_5, b'_0]$ to $A[a_4, a_2]$. Now, by (3), (4), (11), (12), and our previous statements, we can obtain a new
graph $H$ from $G$ by splitting $a_1, a_4$ as $s, s'$ and $t, t'$, respectively, such that $a_1 := s$ in $H$, and $(H, a_0, b_1, a_1, s, t, s', t', a_2, b_2)$ is planar, a contradiction to (iii) of Lemma 6.0.1.

Now, by $u^2_1 \in B[b_5, b'_0]$, $G$ has no edge from $B[b_5, b_2]$ to $A[a_4, a_2]$, and so, by (3), (4), (11), (12), and our previous statements, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 6.0.1. □

Now, we see that $A(a_5, a_4) = \emptyset$. For otherwise, there exists $a \in A(a_5, a_4)$. Since the degree of $a$ in $G$ is at least 6, then combined with (11), $G$ has an edge $e$ from $a$ to $b \in B[b_4, b_5]$ with $b \notin \{b_1, b_5, b_0\}$. Now, if $b \in B(b_4, b_0)$, then $(e_3, e_4, e, e_0, e_5)$ is a 5-edge configuration; if $b \in B(b_0, b_5)$, then $(f^2_2, e_5, e, e_0, e_4)$ is a 5-edge configuration.

We may assume $G$ has no edge from $A(a_1, a_5)$ to $B[b_1, b_4]$; otherwise, such an edge together with $e_0, e_4, e_5$ forms a doublecross.

Then we claim that $G$ has no edge from $B(b_0, b'_0)$ to $A(a_1, a_2)$. For, suppose $G$ has an edge $e$ from $b \in B(b_0, b'_0)$ to $a \in A(a_1, a_2)$. Then $b'_0 \notin B(b_5, b_2)$; otherwise, $(f^2_2, e'_0, b_0, e_5, e_0, e_4)$ is a 5-edge configuration. So $b'_0 \in B(b_4, b_5)$. Hence, $b \in B(b_4, b_5)$, and by (11), $a \in A[a_5, a_4]$. But then, $(f^2_2, e'_0, e, e_0, e_4)$ is a 5-edge configuration.

Now, if $G$ has no edge from $a_4$ to $B(b_4, b_5)$, then, combined with (3), (4), (6), (11), and our previous statements, we can obtain a new graph $H$ from $G$ by splitting $b_4$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1.

So we may assume $G$ has an edge $e$ from $a_4$ to $b \in B(b_4, b_5)$. Then $b \notin B(b_0, b_5)$; otherwise, $(f^2_2, e_5, e, e_0, e_4)$ is a 5-edge configuration. Moreover, $G$ has no edge from $b_4$ to $a_5$, since, otherwise, such an edge together with $e_5, e_0, e$ forms a doublecross. But now, combined with (3), (4), (6), (11), and our previous statements, we can obtain a new graph $H$ from $G$ by splitting $a_4$ as $s, s'$, such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 6.0.1. □

**Lemma 6.0.3** Suppose $\gamma$ is infeasible, and $A', B'$ is a core $a_0$-frame in $\gamma$. Suppose $\mathcal{P} = (e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration w.r.t. an ideal frame $A, B$ in $\gamma$ with $a_1, a_3, a_4, a_2$
on $A$ in order and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ on $B$ in order. Let $G_0 := G - A$, where $(G_0, a_0, b_1, B, b_2)$ is planar.

Then $G_0$ has a separation $(G_1, G_2)$ such that $b_1', b_2' \in V(G_1) \cap V(G_2)$, $|V(G_1) \cap V(G_2)| \leq 3$, $\{a_0, b_1, b_2\} \subseteq V(G_1)$, $B[b_1', b_2'] \subseteq G_2$, $|V(G_1 - G_2)| \geq 1$, and one of the following holds:

(i) $V(G_1) \cap V(G_2) = \{a_0', b_1', b_2', b_1' \in B[b_3, b_4], b_2' \in B[b_7, b_2], \text{ and } G_0 \text{ has a path from } a_0 \text{ to } B(b_1', b_2') \text{ through } a_0' \text{ and internally disjoint from } B$.

(ii) $V(G_1) \cap V(G_2) = \{b_1', b_2', b_1' \in B[b_3, b_4], \text{ and } b_2' \in B[b_7, b_2]$.

(iii) $V(G_1) \cap V(G_2) = \{b_1', b_2', b_1' \in B[b_3, b_4], \text{ and } b_2' \in B[b_6, b_7]$.

(iv) $V(G_1) \cap V(G_2) = \{b_1', b_2', b_1' \in B(b_4, b_5), \text{ and } b_2' \in B[b_7, b_2]$.

Proof. Since otherwise, $G_0 - (B[b_3, b_4] \cup B[b_7, b_2])$ contains disjoint paths $B_1, A_0$ from $b_1, a_0$ to $b_5, b_6$, respectively. Now $(A - A[a_5, a_7]) \cup e_3 \cup B[b_3, b_4] \cup e_4 \cup e_6 \cup A_0$ and $B_1 \cup e_5 \cup A[a_5, a_7] \cup e_7 \cup B[b_7, b_2]$ show that $\gamma$ is feasible, a contradiction. □

Definition. Let $\gamma = (G, a_0, a_1, a_2, b_1, b_2)$ be a rooted graph. Suppose $A, B$ is an ideal frame w.r.t. $a_0$ in $\gamma$, and $(e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration w.r.t. $A, B$, where $e_i = a_i b_i \in E(G)$ with $a_i \in V(A)$ and $b_i \in V(B)$ for $i = 3, 4, 5, 6, 7$.

For notational convenience, we further assume $a_1, a_3, a_4, a_2$ occur on $A$ in order, and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ occur on $B$ in order. Then we say that $(e_3, e_4, e_5, e_6, e_7)$ is an ideal frame if the following requirements are satisfied in the order listed:

- $B[b_4, b_7]$ is maximal,
- $B[b_6, b_7]$ is minimal,
- $B[b_4, b_5]$ is minimal,
- $A[a_5, a_7]$ is minimal,
• $A[a_3, a_4]$ is maximal,

• $B[b_1, b_3]$ is minimal, and

• $A[a_6, a_5] \cap A[a_6, a_7]$ is maximal.

**Lemma 6.0.4** Suppose $\gamma$ is infeasible, and $A', B'$ is a core $a_0$-frame in $\gamma$. Suppose $\mathcal{P} = (e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration w.r.t. an ideal frame $A, B$ in $\gamma$ with $a_1, a_3, a_4, a_2$ on $A$ in order and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ on $B$ in order. Let $G_0 := G - A$, where $(G_0, a_0, b_1, B, b_2)$ is planar.

Assume $a_7 \in A[a_1, a_5], a_6 \in A(a_5, a_2), G$ has no edge from $B(b_4, b_5)$ to $A[a_1, a_5]$, and $G$ has no edge from $B[b_7, b_2]$ to $A(a_5, a_2)$. Then there does not exist a separation $(G_1, G_2)$ in $G_0$ such that $V(G_1 \cap G_2) = \{b_1^*, b_2^*\}$ with $b_1^* \in B[b_1, b_4]$ and $b_2^* \in B[b_6, b_2]$, $\{a_0, b_1, b_2\} \subseteq V(G_1), V(B[b_1^*, b_2^*]) \subseteq V(G_2)$, and $|V(G_1 - G_2)| \geq 1$.

**Proof.** We choose $(G_1, G_2)$ so that $B[b_1^*, b_2^*]$ is maximized. To avoid forming a doublecross with $e_5, e_6$, we may assume

1. $G$ has no parallel edges from $B[b_6, b_2]$ to $A[a_1, a_5]$.

2. If $ab \in E(G)$ with $a \in A(a_5, a_2)$ and $b \in V(B)$ then $b \in B[b_4, b_6]$.

Suppose $e = ab \in E(G)$ with $a \in A(a_5, a_2)$ and $b \in V(B) - V(B[b_4, b_6])$. We may assume $b \in B[b_1, b_4]$. For, if $b \in B[b_6, b_2]$, then since $G$ has no edge from $B(b_7, b_2)$ to $A(a_5, a_2), b \in B[b_6, b_7]$. Now $(e_3, e_4, e_5, e, e_7)$ contradicts the choice of $\mathcal{P}$.

Suppose $a \in A(a_5, a_4)$. Then $b \in B[b_3, b_4]$ to avoid the doublecross $e, e_3, e_4, e_5$. But then $b_3 \neq b_4$, and $(e_3, e, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Thus $a \in A[a_4, a_2]$. Then $b = b_1$ as, otherwise, $(e_3, e, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $a \neq a_2$; and let $e' = a_2b' \in E(G)$ with $b' \in V(B) - \{b_1, b_2\}$. Then $b' \in B[b_7, b_2]$ to avoid the doublecross $e, e', e_3, e_7$. But it contradicts that $G$ has no edge from $B[b_7, b_2]$ to $A(a_5, a_2)$. $\square$
Let $e_9 = a_9b_9$ with $a_9 \in A[a_1, a_5]$ and $b_9 \in B(b_1^*, b_2^*)$, and choose $e_9$ so that $A[a_1, a_9]$ is minimal.

Since $G^*$ is 6-connected, then

(3) There exists $e^* = a^*b^* \in E(G)$ with $a^* \in A(a_9, a_2)$ and $b^* \in B - B[b_1^*, b_2^*]$.

By (2), we have $a^* \notin A(a_5, a_2)$; so $a^* \in A(a_9, a_5)$ and $a_9 \in A[a_1, a_5]$. Moreover,

(4) $b_9 \in B(b_1^*, b_4) \cup B[b_6, b_2^*]$.

For otherwise, $b_9 \in B(b_5, b_6)$ by $a_9 \in A[a_1, a_5]$ and the assumption that $G$ has no edge from $B(b_4, b_5)$ to $A[a_1, a_5]$. Now, $a_9 \in A[a_7, a_5]$ to avoid the doublecross $e_5, e_6, e_7, e_9$. By $a^* \in A(a_9, a_5)$, $b^* \in B[b_1^*, b_4^*]$ to avoid the doublecross $e_9, e^*, e_5, e_6$, and $b^* \notin B[b_1, b_3^*]$ to avoid the doublecross $e_3, e^*, e_6, e_7$. Hence, $b_3, b^* \in B(b_1, b_4)$, and $(e_3, e^*, e_9, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. $\square$

**Case 1.** $b_9 \in B[b_6, b_2^*]$.

Then $b^* \in B[b_1^*, b_4^*]$ to avoid the doublecross $e_9, e^*, e_5, e_6$.

We claim that $G$ has no edge from $B(b_1^*, b_4)$ to $A[a_1, a_5]$. For suppose $e = ab \in E(G)$ with $a \in A[a_1, a_5]$ and $b \in B[b_1^*, b_4]$. Note that $b_1^*$ and $b_2^*$ are feet of some connector $J$, and $B[b_1^*, b_2^*] \subseteq J$. Let $u_1, u_2$ denote the extreme hands for $J$. Note that $e^*$ is from $A(x_1, x_2)$ to $B[b_1^*, b_4^*]$; so we know $(J - b_1^*, u_1, A(u_1, u_2), u_2, b_2^*)$ is planar by Lemma 3.0.4. But this cannot be the case because of $e, e_4, e_5$.

Because of $(G_1, G_2)$, $G$ has a separation $(G_1', G_2')$ such that $V(G_1' \cap G_2') = \{b_1', b_2'\}$ with $b_1', b_1, b_4, b_6, b_2, b_2'$ on $B$ in order, $B[b_1', b_2'] \subseteq G_1'$, and $\{a_0, b_1', b_2'\} \subseteq V(G_2')$. We choose $(G_1', G_2')$ such that $B[b_6, b_2']$ is minimal and, subject to this, $B[b_1', b_1']$ is minimal.

Let $e_9' = a_9'b_9' \in E(G)$ with $a_9' \in A[a_1, a_5]$ and $b_9' \in B(b_1', b_2')$, and choose $e_9'$ so that $A[a_1, a_9']$ is minimal. We may assume that there exists $e' = a'b' \in E(G)$ with $a' \in A(a_9', a_2]$ and $b' \in B - B[b_1', b_2']$ (as $G^*$ is 6-connected).

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Then \( b'_0 \in B[b_6, b'_2] \) (since \( G \) has no edge from \( B(b'_1, b_4) \) to \( A[a_1, a_5] \)) and \( b' \in B[b_1, b'_1] - b'_1 \) (to avoid doublecross with \( e_5, e_6, e'_9, e' \)). So \((e'_9, e_6, e_5, e'_4)\) is a 5-edge configuration.

By Lemma 2.0.9 and 6.0.3, \( G_0 \) has a cut that contradicts the choice of \((G_1, G_2)\) or \((G'_1, G'_2)\).

**Case 2.** \( b_0 \in B(b'_1, b_4) \).

Then \( b^* \in B(b^*_2, b_2) \) to avoid the doublecross \( e_9, e^*, e_4, e_5 \).

We claim that \( G \) has no edge from \( B[b_6, b^*_2] \) to \( A[a_1, a_5] \). For suppose \( e = ab \in E(G) \) with \( a \in A[a_1, a_5] \) and \( b \in B[b_6, b^*_2] \). Note that \( b'_1 \) and \( b^*_2 \) are feet of some connector \( J \), and \( B[b'_1, b^*_2] \subseteq J \). Let \( u_1, u_2 \) denote the extreme hands for \( J \). Note that \( e^* \) is from \( A(u_1, u_2) \) to \( B(b^*_2, b_2) \); so we know \((J - b^*_2, u_1, A(u_1, u_2), u_2, b^*_1)\) is planar by Lemma 3.0.4. But this cannot be the case because of \( e, e_5, e_6 \).

Because of \((G_1, G_2)\), \( G \) has a separation \((G'_1, G'_2)\) such that \( V(G'_1 \cap G'_2) = \{b'_1, b'_2\} \) with \( b'_1, b'_2, b_4, b_6, b_2, b_2 \) on \( B \) in order, \( B[b'_1, b'_2] \subseteq G'_1 \), and \( \{a_0, b'_1, b'_2\} \subseteq V(G'_2) \). We choose \((G'_1, G'_2)\) such that \( B[b'_1, b_4] \) is minimal and, subject to this, \( B[b'_2, b^*_2] \) is minimal.

Let \( e'_9 = a'_9 b'_9 \in E(G) \) with \( a'_9 \in A[a_1, a_5] \) and \( b'_9 \in B(b'_1, b'_2) \), and choose \( e'_9 \) so that \( A[a_1, a'_9] \) is minimal. We may assume that there exists \( e' = a'b' \in E(G) \) with \( a' \in A(a'_9, a_2) \) and \( b' \in B - B[b'_1, b'_2] \) (as \( G^* \) is 6-connected).

Then \( b'_0 \in B(b'_1, b_4) \) (since \( G \) has no edge from \( B[b_6, b^*_2] \) to \( A[a_1, a_5] \)) and \( b' \in B[b'_2, b_2] - b'_2 \) (to avoid doublecross \( e'_4, e'_9, e_4, e_5 \)). So \((e'_9, e_4, e_5, e_6, e'_4)\) is a 5-edge configuration. By Lemma 2.0.9 and 6.0.3, \( G_0 \) has a separation that contradicts choice of \((G_1, G_2)\) or \((G'_1, G'_2)\).

\( \square \)

**Lemma 6.0.5** Suppose \( \gamma \) is infeasible, and \( A', B' \) is a core \( a_0 \)-frame in \( \gamma \). Suppose \( \mathcal{P} = (e_3, e_4, e_5, e_6, e_7) \) is a 5-edge configuration w.r.t. an ideal frame \( A, B \) in \( \gamma \) with \( a_1, a_3, a_4, a_2 \) on \( A \) in order and \( b_1, b_3, b_4, b_5, b_6, b_7, b_2 \) on \( B \) in order. Let \( G_0 := G - A \), where \((G_0, a_0, b_1, B, b_2)\) is planar.

Then \( G_0 \) has a separation \((G_1, G_2)\) such that \( b'_1, b'_2 \in V(G_1) \cap V(G_2) \), \(|V(G_1) \cap V(G_2)| \leq 3\), \(|V(G_1) - V(G_2)| \geq 1\), \( \{a_0, a_1, a_2\} \subseteq V(G_1) \), \( B[b'_1, b'_2] \subseteq G_2 \), and one of the following holds:

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(i) $V(G_1) \cap V(G_2) = \{a'_0, b'_1, b'_2\}, b'_1 \in B[b_1, b_4], b'_2 \in B[b_7, b_2]$, and $G_0$ has a path from $a_0$ to $B(b'_1, b'_2)$ through $a'_0$ and internally disjoint from $B$.

(ii) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}, b'_1 \in B[b_1, b_4], \text{ and } b'_2 \in B[b_7, b_2]$. 

(iii) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}, b'_1 \in B[b_1, b_4], b'_2 \in B[b_6, b_7]$, and $G$ has no edge from $B(b'_2, b_7)$ to $A - a_7$.

(iv) $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}, b'_1 \in B(b_4, b_5), b'_2 \in B[b_7, b_2]$, and $G$ has no edge from $B(b_4, b'_1)$ to $A - a_4$.

Proof. By Lemma 6.0.3, $G_0$ has a separation $(G_1, G_2)$ such that $V(G_1) \cap V(G_2) = \{b'_1, b'_2\}$, $B[b'_1, b'_2] \subseteq G_1$, $\{a_0, b_1, b_2\} \subseteq G_2$, and one of the following holds:

(A) $b'_1 \in B[b_1, b_4], b'_2 \in B[b_6, b_7]$, and there exists $e_8 = a_8b_8 \in E(G)$ with $a_8 \in V(A - a_7)$ and $b_8 \in V(B(b'_2, b_7))$, or

(B) $b'_1 \in B(b_4, b_5), b'_2 \in B[b_7, b_2]$, and there exists $e_8 = a_8b_8 \in E(G)$ with $a_8 \in V(A - a_4)$ and $b_8 \in V(B(b_4, b'_1))$.

So we consider two cases.

Case 1. (A) holds.

We choose $\{b'_1, b'_2\}$ so that $B[b'_1, b_4]$ is minimal and, subject to this, $B[b'_2, b_7]$ is minimal. We also choose $e_8$ so that $A[a_8, a_5]$ is minimal. Note that $a_8 \in A[a_5, a_7]$, for otherwise, $(e_3, e_4, e_5, e_8, e_7)$ contradicts $\mathcal{P}$. So $a_5 \neq a_7$.

We consider two subcases according to the positions of $a_5$ and $a_7$.

Subcase 1.1. $a_5 \in A[a_7, a_2]$.

First, we note that for $e = ab \in E(G)$ with $a \in V(A)$ and $b \in V(B)$, if $a \in A(a_5, a_2)$ and $b \in B(b_1, b'_1)$, then $(e_3, e, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$, and if $a \in A(a_8, a_5)$ or $b \in B[b_1, b_3]$ then $e, e_3, e_8, e_4$ form a doublecross. So we have
(1.1.1) $G$ has no edge from $A(a_5, a_2)$ to $B(b_1, b'_1)$, and $G$ has no edge from $A(a_8, a_5)$ to $B(b_1, b_3)$.

(1.1.2) $G$ has no edge from $A(a_8, a_2)$ to $B(b'_2, b_2) + b_1$.

For, let $e = ab$ with $a \in A(a_8, a_2)$ and $b \in B(b'_2, b_2) + b_1$.

Suppose $b = b_1$. Then $a \neq a_2$, and let $e_2 = a_2b' \in E(G)$ with $b' \in B(b_1, b_2)$. Then $b' \in B[b_7, b_2)$ to avoid the doublecross $e, e_3, e_7, e_2$. But now $(e_2, e_7, e_5, e_3, e)$ contradicts the choice of $\mathcal{P}$.

So $b \in B(b'_2, b_2)$. If $b \in B(b'_2, b_7)$ then $a \in A(a_5, a_2)$ (by the minimality of $A[a_8, a_5]$); but then $(e_3, e_4, e_5, e, e_7)$ contradicts the choice of $\mathcal{P}$. So $b \in B[b_7, b_2)$. Then $a \in A(a_5, a_2)$, as otherwise $(e_3, e_4, e_5, e_6, e)$ contradicts the choice of $\mathcal{P}$. Now $(e, e_7, e_8, e_6, e_5)$ is a 5-edge configuration. By Lemma 2.0.9 and 6.0.3, $G_0$ has a separation, which admits (i) or (ii), or contradicts the choice of $\{b'_1, b'_2\}$. □

(1.1.3) $G$ has no edge from $A(a_7, a_2)$ to $b_2$.

To prove (1.1.3), let $e = ab_2 \in E(G)$ with $a \in A(a_7, a_2)$. We claim that $a \in A(a_5, a_4)$. To see this, first note that $a \neq a_2$. Moreover, $a \in A(a_5, a_2)$; as otherwise, $(e_3, e_4, e_5, e_6, e)$ contradicts the choice of $\mathcal{P}$. Now suppose to the contrary that $a \notin A(a_5, a_4)$. Then $a \in A[a_4, a_2)$, and let $e_2 = a_2b'_2 \in E(G)$ with $b'_2 \in V(B) - \{b_1, b_2\}$. So $b'_2 \in B(b_1, b_4)$ to avoid the doublecross $e_2, e, e_4, e_8$. But then $(e_3, e_2, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Thus $b_7 = b_2$, or else $(e_3, e_4, e_5, e_7, e)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_8 = a_5$, or else $(e_3, e_4, e_5, e_8, e)$ contradicts the choice of $\mathcal{P}$.

Suppose $a_6 \in A[a_1, a_7)$. Let $e'_7 = a_7b'_7 \in E(G)$ with $b'_7 \in V(B - b_7)$. Then $b'_7 \notin B[b_1, b_5)$ to avoid the doublecross $e_6, e'_7, e_7, e_8$. Also $b'_7 \neq b_6$ as otherwise $(e_3, e_4, e'_7, e_8, e_7)$ contradicts the choice of $\mathcal{P}$. So $b'_7 \in B(b_6, b_2)$. Then $(e_3, e_4, e_5, e'_7, e_7)$ contradicts the choice of $\mathcal{P}$.

So $a_6 \in A(a_5, a_2)$ for all choices of $e_6$. Then $a_6 \in A[a_4, a_2)$, or else $(e_3, e_4, e_6, e_8, e)$ contradicts the choice of $\mathcal{P}$. Let $e' = ab' \in E(G)$ with $b' \in V(B - b_2)$. Then $b' \neq b_6$ as
\( a_6 \in A[a_1, a_2] \) for all choices of \( e_6 \). So \( b' \in B(b_6, b_2) \) to avoid the doublecross \( e_8, e_6, e, e' \).

But then \((e_3, e_4, e_5, e', e_7)\) contradicts the choice of \( \mathcal{P} \). \( \square \)

(1.1.4) There exists \( e_9 = a_9b_9 \in E(G) \) with \( a_9 \in A[a_1, a_8] \) and \( b_9 \in B(b_1', b_2'] \).

For, suppose such an edge does not exist. Then \( a_6 \in A(a_5, a_2] \) and \( G \) has no edge from \( B(b_4, b_5) \) to \( A(a_1, a_5] \) by the choice of \( \mathcal{P} \). Note that we have \( a_5 \neq a_7 \) and \( a_7 \in A[a_1, a_5] \) and that, by (1.1.2) and (1.1.3), \( G \) has no edge from \( B(b_7, b_2) \) to \( A(a_5, a_2] \). Thus, we may apply Lemma 6.0.4, a contradiction. \( \square \)

(1.1.5) \( b_9 \in B(b_4, b_2'] \), \( a_9 = a_3 \), and so all edges from \( B(b_1', b_2'] \) to \( A[a_1, a_8] \) must be from \( B(b_4, b_2'] \) to \( a_3 \).

First, suppose \( b_9 \in B(b_1', b_4] \). Then \((e_9, e_4, e_5, e_6, e_8)\) is 5-edge configuration. Thus, by Lemma 2.0.9 and 6.0.3, \( G_0 \) has a separation, which admits (i) or (ii), or contradicts the choice of \( \{b_1', b_2\} \). So we may assume \( b_9 \in B(b_4, b_2'] \). Suppose \( a_9 \neq a_3 \). Then \( a_9 \in A(a_3, a_4) \), to avoid the doublecross \( e_3, e_9, e_5, e_7 \). But now \((e_3, e_4, e_9, e_8, e_7)\) is a 5-edge configuration contradicting the choice of \( \mathcal{P} \). \( \square \)

(1.1.6) \( a_4 = a_2 \).

Suppose \( a_4 \neq a_2 \). Let \( e_2 = a_2b_2^* \in E(G) \) with \( b_2^* \in V(B) \). Then \( b_2^* \in B(b_1, b_4] \) to avoid the doublecross \( e_2^*, e_4, e_9, e_8 \). Now \((e_3, e, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \). \( \square \)

Thus, \( G \) has no edge \( e \) from \( B(b_1, b_1') \) to \( v \in V(A(a_8, a_2]) \); for, if \( v \neq a_2 \) then \( e, e_9, e_8, e_4 \) would form a doublecross, and if \( v = a_2 \) then \((e_3, e, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \). Hence, by (1.1.2) and (1.1.5), \( G \) has a 5-separation \((H_1, H_2)\) such that \( V(H_1 \cap H_2) = \{b_1', b_2', a_8, a_3, a_2\} \), \( V(A(a_8, a_2]) \cup V(B(b_1', b_2')] \cup \{a_3\} \subseteq V(H_1) \), and \( V(A[a_3, a_8]) \cup \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_2) \), a contradiction.

Subcase 1.2. \( a_5 \in A[a_1, a_7] \).

Then \( a_6 \notin A(a_4, a_2] \) to avoid the doublecross \( e_4, e_6, e_5, e_7 \), and \( a_6 \notin A(a_7, a_4) \) as, otherwise, \((e_3, e_4, e_6, e_8, e_7)\) contradicts the choice of \( \mathcal{P} \). Hence,
(1.2.1) $a_6 \in A[a_1, a_5]$ or $a_6 = a_4$.

(1.2.2) There exists $v \in \{a_4, b_4\}$ such that all edges from $A(a_8, a_2)$ to $B(b'_1, b'_2)$ are incident with $v$.

To prove (1.2.2), we first claim that $G$ has no edge from $A(a_8, a_2) - a_4$ to $B(b'_1, b'_2) - b_4$. For otherwise, suppose there exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A(a_8, a_2) - a_4$ to $b_9 \in B(b'_1, b'_2) - b_4$. If $b_9 \in B(b'_1, b_4)$ then $a_9 \in A(a_4, a_2)$ to avoid the doublecross $e_9, e_4, e_7, e_8$; so $(e_3, e_9, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $b_9 \in B(b_4, b'_2)$. Then $a_9 \in A(a_8, a_4)$ to avoid the doublecross $e_4, e_9, e_8, e_7$. Now $(e_3, e_4, e_9, e_8, e_7)$ contradicts the choice of $\mathcal{P}$.

Next, we see that either no edge is from $b_4$ to $A(a_8, a_2) - a_4$, or no edge is from $a_4$ to $B(b'_1, b'_2) - b_4$. In fact, by the choice of $\mathcal{P}$, any edge from $b_4$ to $A(a_8, a_2) - a_4$ must end in $A(a_8, a_4)$, and any edge from $a_4$ to $B(b'_1, b'_2) - b_4$ must end in $B(b_4, b'_2)$. If $G$ has an edge from $b_4$ to $A(a_8, a_2) - a_4$ and an edge from $a_4$ to $B(b'_1, b'_2) - b_4$, then such two edges and $e_7, e_8$ form a doublecross in $\gamma$, a contradiction. □

Define $a'_1 \in A[a_1, a_8]$ such that $G$ has no edge from $A[a_1, a'_1)$ to $B(b'_1, b'_2)$ and, subject to this, $A[a_1, a'_1]$ is maximal. By the definition of $a'_1$, $G$ has an edge $e_1$ from $a'_1$ to $b \in B(b'_1, b'_2)$.

We claim that $a'_1 \in A[a_3, a_8)$. For otherwise, $a'_1 \in A[a_1, a_3)$. Now, if $b \in B(b_3, b'_2)$ then $e_1, e_3, e_4, e_8$ form a doublecross; if $b \in B(b_1, b_3)$ then $(e_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

So $a_1, a_3, a'_1, a_5, a_8, a_7, a_4, a_2$ occur on $A$ in order.

(1.2.3) $G$ has no edge from $A(a'_1, a_8)$ to $B - B[b'_1, b'_2]$.

For, otherwise, $a'_1 \neq a_8$, and there exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A(a'_1, a_8)$ to $b_9 \in B - B[b'_1, b'_2]$. Then $b_9 \notin B[b_1, b'_1)$ to avoid the doublecross $e_1, e_9, e_4, e_7$.

We claim $b_9 = b_2$ and $a_9 \notin A[a_5, a_8)$. For, if $b_9 \in B(b'_2, b_7)$ then $a_9 \in A(a'_1, a_5)$ by the choice of $e_8$ (that $A[a_5, a_8]$ is minimal); now $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $b_9 \in B[b_7, b_2]$. Thus, $a_9 \notin A[a_5, a_8)$; as otherwise $(e_3, e_4, e_5, e_8, e_9)$ contradicts the
choice of $P$. Now suppose $b_9 \neq b_2$. Then $(e_7, e_9, e_8, e_5)$ is a 5-edge configuration. Thus, by Lemma 2.0.9 and 6.0.3, $G_0$ has a separation, which admits (i) or (ii), or contradicts the choice of $\{b_1', b_2'\}$.

We see $a_8 = a_5$; otherwise, $(e_3, e_4, e_5, e_8, e_9)$ contradicts the choice of $P$. Moreover, $a_4 = a_2$; for otherwise, $G$ has an edge $e'$ from $a_2$ to $B$, then either $(e_3, e', e_5, e_6, e_7)$ contradicts the choice of $P$ or $e', e_4, e_5, e_7$ form a doublecross.

Next, we claim that all edges from $A(a_8, a_2)$ to $B$ must end in $\{b_1, b_2\}$. First, $G$ has no edge from $A(a_8, a_2)$ to $b_1$; otherwise, such an edge together with $e_7, e_3, e_4$ forms a doublecross. $G$ has no edge from $A(a_8, a_2)$ to $B(b_1, b_4)$; otherwise, such an edge together with $e_3, e_5, e_1, e_9$ forms a 5-edge configuration contradicting the choice of $P$. $G$ has no edge from $A(a_8, a_2)$ to $B(b_1, b_8)$; otherwise, such an edge together with $e_3, e_4, e_8, e_7$ forms a 5-edge configuration contradicting the choice of $P$. $G$ has no edge from $A(a_8, a_2)$ to $B(b_2, b_8)$; otherwise, such an edge together with $e_3, e_4, e_5, e_9$ forms a 5-edge configuration contradicting the choice of $P$.

Therefore, since $a_7 \in A(a_8, a_2)$, then $\{a_2, a_8, b_2, b_4\}$ is a 4-cut in $G$ separating $a_7$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

By (1.2.2) and (1.2.3), $G$ has a separation $(H_1, H_2)$ such that $V(H_1 \cap H_2) = \{b_1', b_2', a_8, a_1', v\}$, $b_5 \in V(H_2 - H_1)$, and $\{a_0, a_1, a_2, b_1, b_2\} \subseteq H_1$, a contradiction.

Case 2. (B) holds.

We choose $(G_1, G_2)$ such that $B[b_1', b_2']$ is maximal.

We claim that $a_8 \in A[a_1, a_3] \cup A(a_4, a_2)$. For, suppose $a_8 \in A(a_3, a_4)$. Then $a_6 \in A[a_7, a_8]$ and $a_8 \notin A[a_7, a_5]$; for otherwise $(e_3, e_4, e_8, e_6, e_7)$ contradicts the choice of $P$.

Therefore, $a_5 \notin A[a_6, a_8]$ (since $a_6 \notin A[a_5, a_7]$). So $(e_3, e_4, e_5, e_6)$ is a 5-edge configuration. Thus, by Lemma 2.0.9 and 6.0.3, $G_0$ has a separation, which admits (i) or (ii), or contradicts the choice of $\{b_1', b_2'\}$.

Therefore, we have distinguished two cases.

Subcase 2.1. $a_8 \in A(a_4, a_2)$.
Choose \( e_8 \) so that \( A[a_8, a_2] \) is minimal. Note that \( a_6 \in A[a_8, a_2] \) and \( a_7 \in A(a_3, a_5) \), since, otherwise, \( e_4, e_8 \) and two of \( \{e_5, e_6, e_7\} \) force a doublecross.

(2.1.1) \( G \) has no edge from \( A(a_5, a_2) \) to \( B[b_1, b_4] \cup B(b_6, b_2) \).

For, let \( e = ab \in E(G) \) with \( a \in A(a_5, a_2) \) and \( b \in B[b_1, b_4] \cup B(b_6, b_2) \).

Suppose \( b \in B(b_6, b_2) \). Then \( a \in A[a_8, a_2] \) to avoid the doublecross \( e, e_4, e_5, e_8 \). So \( b \in B[b_7, b_2] \), or else \( (e_3, e_4, e_5, e, e_7) \) contradicts the choice of \( \mathcal{P} \). Suppose \( b = b_2 \). Then \( a \neq a_2 \), so there exists \( e' = a_2b' \in E(G) \) with \( b' \in B(b_1, b_2) \). Now \( b' \in B(b_1, b_4) \) to avoid the doublecross \( e_4, e_5, e, e' \). But then, \( (e_3, e', e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). Thus, \( b \neq b_2 \). Now \( (e, e_7, e_5, e_8, e_4) \) is a 5-edge configuration. Hence, by Lemma 2.0.9 and 6.0.3, \( G_0 \) has a separation, which admits (i) or (ii), or contradicts the choice of \( \{b_1', b_2'\} \).

Thus, \( b \in B[b_1, b_4] \) for every choice of \( e = ab \). Assume \( a \in A[a_4, a_2] \). Then \( b = b_1 \), or else \( (e, e_3, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). Now, since \( G \) has no edge from \( B(b_6, b_2) \) to \( A(a_5, a_2) \), \( G \) has an edge from \( a_2 \) to \( B(b_1, b_7) \), which together with \( e, e_3, e_7 \) forms a doublecross. So \( a \in A(a_5, a_4) \). Then either \( e_3, e_4, e_5, e \) form a doublecross, or \( (e_3, e, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). \( \square \)

(2.1.2) \( G \) has no edge from \( B(b_1, b_3) \) to \( A \).

For otherwise, let \( e = ab \in E(G) \) with \( a \in A \) and \( b \in B(b_1, b_3) \). If \( a \in A[a_1, a_3] \), then \( (e, e_4, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \); if \( a \in A(a_3, a_4) \), then \( e, e_3, e_4, e_7 \) form a doublecross; if \( a \in A[a_4, a_2] \), then \( (e_3, e, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). Hence, (2.1.2) holds. \( \square \)

(2.1.3) \( b_2' = b_2 \) and \( G_0 \) has a separation \( (G_1', G_2') \) that \( V(G_1' \cap G_2') = \{b_1, b_2', a_0\} \), \( b_2'' \in B(b_1', b_2'), B[b_1, b_2'] \subseteq G_1', \) and \( \{a_0, b_1, b_2\} \subseteq V(G_2') \).

For, otherwise, we claim that there exists \( v \in \{a_5, b_3\} \) such that all edges from \( B(b_1', b_2') \) to \( A[a_1, a_8] \) in \( G \) contain \( v \). To prove this, we first assume that \( b_5 \in B(b_1', b_2') \), and there
exist \( e'_5 = a_5 b'_5, e''_5 = a'_5 b_5 \in E(G) \) with \( a'_5 \in A[a_1, a_8] \) and \( b'_5 \in B(b'_1, b'_2) \) such that \( a'_5 \neq a_5 \) and \( b'_5 \neq b_5 \). Then \( e'_5, e''_5 \) form a cross to avoid the doublecross \( e'_5, e''_5, e_4, e_8 \). But now, \( b'_5 \in B(b_5, b'_2) \) by the choice of \( \mathcal{P} \), and so \( (e_6, e'_5, e''_5, e_8, e_4) \) is a 5-edge configuration; and since (i) or (ii) does not hold, then (2.1.3) follows from Lemmas 2.0.9 and 6.0.3 and the choice of \( \{b'_1 b'_2\} \). So assume that such \( e'_5, e''_5 \) do not exist. Therefore, if the claimed \( v \) does not exist, then there exists \( e = ab \in E(G) \) such that \( a \in A[a_1, a_8] - a_5 \) and \( b \in B(b'_1, b'_2) - b_5 \). If \( b \in B(b'_1, b_5) \) then \( a \in A(a_5, a_8) \) to avoid the doublecross \( e, e_5, e_4, e_8 \).

Hence, \((e_6, e_5, e, e_8, e_4)\) is a 5-edge configuration and (2.1.3) follows from Lemmas 2.0.9 and 6.0.3 and the choice of \( \{b'_1 b'_2\} \); or else, (i) or (ii) holds. So \( b \in B(b_5, b'_2) \). Then \( a \notin A(a_5, a_8) \) to avoid the doublecross \( e_4, e_5, e_8, e \). Hence, \((e, e_6, e_5, e_8, e_4)\) is a 5-edge configuration and (2.1.3) follows from Lemmas 2.0.9 and 6.0.3 and the choice of \( \{b'_1 b'_2\} \); or else, (i) or (ii) holds.

Now, we see that \( G \) has no edge from \( A(a_8, a_2) \) to \( B - B[b'_1, b'_2] \). For otherwise, by our claim, \( G \) has an edge \( e = ab \) with \( a \in A(a_8, a_2) \) and \( b \in B - [b'_1, b'_2] \). By (2.1.1), \( b \in B(b_4, b'_1) \). By the choice of \( \mathcal{P} \), \( b \neq b_4 \). So \( b \in B(b_4, b'_1) \), which contradicts the choice of \( e_8 \).

Thus, \( G \) has a separation \((H_1, H_2)\) such that \( V(H_1 \cap H_2) = \{v, a_8, a_2, b'_1, b'_2\} \) and \( b_0 \in V(H_2 - H_1) \), a contradiction. \( \square \)

By (2.1.3), \( \alpha(A, B) \leq 1 \). We may assume

\[ (2.1.4) \quad b''_2 \notin B[b_7, b_2], \]

and either \( b_7 = b_2 \) (in which case let \( B_0 = B[b''_2, b_2] \)) or \( b_7 \neq b_2 \) and

\[ G_0 - (B[b_1, b''_2] \cup B[b_7, b_2]) \]

has a path \( B_0 \) from \( b''_2 \) to \( b_2 \).

Clearly, \( b''_2 \notin B[b_7, b_2] \) as otherwise the conclusion of the lemma holds. Now suppose \( b_7 \neq b_2 \) and the desired path \( B_0 \) in \( G_0 - (B[b_1, b''_2] \cup B[b_7, b_2]) \) does not exist. Then there exist \( b''_2 \in V(B[b_7, b_2]) \) and a separation \((H_1, H_2)\) in \( G_0 \) such that \( V(H_1 \cap H_2) = \{b_1, b''_2, a_0\} \).

This implies the conclusion of this lemma, a contradiction. \( \square \)

We choose \( b''_2 \) so that \( B[b''_2, b_7] \) is minimal.
(2.1.5) $G$ has two non-incident edges from $B(b'_1, b'_2)$ to $A[a_1, a_5]$. For otherwise, $b'_1 = b_5$, and there exists $v \in \{a_7, b_7\}$ such that all edges in $G$ from $B(b'_1, b'_2)$ to $A[a_1, a_5]$ are incident with $v$.

$G$ has no edge from $B(b'_1, b_6)$ to $A(a_5, a_8)$. For otherwise, let $e = ab \in E(G)$ with $b \in B(b'_1, b_6)$ and $a \in A(a_5, a_8)$. Now, since $b'_1 = b_5$, then $e, e_4, e_5, e_8$ form a doublecross, a contradiction.

$G$ has no edge from $A(a_8, a_2)$ to $B(b_4, b'_1)$. For otherwise, let $e = ab \in E(G)$ with $b \in B(b_4, b'_1)$ and $a \in A(a_8, a_2)$. By the choice of $e_8$, $b \notin B(b_4, b'_1)$, and so $b = b_4$. But then, $(e_3, e, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Now, combined with (2.1.1), $G$ has a separation $(H_1, H_2)$ of order 5, such that $V(H_1 \cap H_2) = \{v, b'_1, b_2, a_8, a_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(B'[b'_1, b'_2] \cup A[a_8, a_2]) \subseteq V(H_2)$, a contradiction. □

Note that no two edges of $G$ from $B(b'_1, b'_2)$ to $A[a_1, a_4]$ can be parallel, as such edges would form a doublecross with $e_4, e_8$. Therefore, there exist two non-incident edges $e_9 = a'_9 b'_9, e_9'' = a''_9 b''_9$ with $a'_9, a''_9 \in A[a_1, a_5]$ and $b'_9, b''_9 \in B(b'_1, b'_2)$ such that $b_1, b'_9, b''_9$ occur on $B$ in order, and $a_1, a'_9, a''_9$ occur on $A$ in order. Moreover, we further choose $e'_9, e''_9$ so that $A[a'_9, a_2] \cup B[b'_9, b'_2]$ is minimal. By the existence of $e_7$, we have $a'_9 \in A[a_7, a_2]$ and $b''_9 \in B[b_7, b_2]$.

(2.1.6) $G$ has two parallel edges $e', e''$ from $b', b'' \in B(b_3, b'_1)$ to $a', a'' \in A[a_4, a_2]$, with $b_1, b', b'', b_2$ occurring on $B$ in order.

Suppose it fails. Then $b_3 = b_4$ as otherwise $e_4, e_8$ give the desired edges for (2.1.6). Let $e = a_1 b \in E(G)$ with $b \notin \{b_1, b_2, b_3, b_7\}$. Then $b \notin B(b_1, b_3)$; otherwise, $(e, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Moreover, $b \notin B(b_3, b_7)$ to avoid the doublecross $e, e_4, e_7, e_8$. So $b \in B(b_7, b_2)$.

Now, since $(e, e_6, e_5, e_8, e_4)$ is a 5-edge configuration, then $b''_2 \in B[b_6, b_7]$; or else, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of $b'_1, b'_2$ or contradicts
the choice of \( b''_2 \).

Now, let \( a^* \in A[a_1, a_2] \), such that \( G \) has an edge \( e^* \) from \( b^* \in B(b''_2, b_7) \) to \( a^* \), subject to this, \( A[a^*, a_2] \) is minimal, and subject to this, \( B[b''_2, b^*] \) is minimal.

We claim that \( a^* \notin A(a_5, a_2) \). For otherwise, suppose \( a^* \in A(a_5, a_2) \). Now, if \( b^* \in B(b''_2, b_7) \), then \((e_3, e_4, e_5, e^*, e_7)\) is a 5-edge configuration contradicting the choice of \( \mathcal{P} \). So \( b^* \in B(b_7, b_2) \). If \( a^* \in A(a_5, a_8) \), then \( e_4, e_5, e_8, e^* \) form a doublecross; if \( a^* \in A[a_8, a_2] \), then \((e, e^*, e_5, e_8, e_4)\) is a 5-edge configuration contradicting the choice of \( \mathcal{P} \). This finishes our claim.

Now, we further claim that \( G \) has no edge from \( A(a_1, a^*) \) to \( B(b_1, b_3) \cup B(b_3, b''_2) \). For otherwise, let \( e' = a'b' \in E(G) \) with \( a' \in A(a_1, a^*) \) and \( b' \in B(b_1, b_3) \cup B(b_3, b''_2) \). Then \( b' \notin B(b_3, b''_2) \) to avoid the doublecross \( e_4, e_8, e', e^* \). So \( b' \in B(b_1, b_3) \). But then \( a' \notin A(a_3, a^*) \) to avoid the doublecross \( e_3, e_4, e', e_7 \). So \( a' \in A[a_1, a_3] \), and \((e', e_4, e_5, e_6, e_7)\) is a 5-edge configuration contradicting the choice of \( \mathcal{P} \).

We may assume \( G \) has an edge \( e'_7 \) from \( b_7 \) to \( a'_7 \in A(a^*, a_2) \) and an edge \( e'_3 \) from \( b_3 \) to \( a'_3 \in A(a_1, a^*) \). For otherwise, \( G \) has a separation \((H_1, H_2)\) of order 5, such that \( V(H_1 \cap H_2) = \{a_1, a^*, v, b''_2, b_2\} \), \( v \in \{b_3, b_7\} \), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1) \), and \( V(A[a_1, a^*] \cup B[b''_2, b_2]) \subseteq V(H_2) \), a contradiction.

Now, \( G \) has a separation \((H_1, H_2)\) of order 6, such that \( V(H_1 \cap H_2) = \{a_1, a^*, b_3, b''_2, b_7, b_2\} \), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1) \), and \( V(A[a_1, a^*] \cup B[b''_2, b_2]) \subseteq V(H_2) \).

We see that any two edges from \( A[a_1, a^*] \) to \( B[b''_2, b_2] \) are not parallel; or else, such two edges together with \( e_4, e_8 \) form a doublecross. Moreover, by the choice of \( \mathcal{P} \), we can further assume \( a'_7 \in A(a_5, a_2) \).

Now, assume \( b^* \notin B(b''_2, b_7) \). Then since any two edges from \( A[a_1, a^*] \) to \( B[b''_2, b_2] \) are not parallel, then, combined with the choice of \( e^* \), we have \((H_2, a_1, b_3, a^*, b_7, b''_2, b_2)\) is planar, a contradiction to Lemma 2.0.3.

So \( b^* \in B(b''_2, b_7) \). But then \((e'_7, e, e^*, e_6, e'_3)\) is a 5-edge configuration. Now, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of \( b''_2 \). □
We choose \( e', e'' \) in (2.1.5) such that \( B[b_3, b'] \) is minimal and, subject to this, \( B[b'', b'_1] \) is minimal.

Suppose \( G_0 - B(b_1, b_3) - B(b'_1, b_2) \) has disjoint paths \( P_1, P_2 \) from \( b_1, a_0 \) to \( b', b'' \), respectively. Let \( A' := P_2 \cup e'' \cup A[a'', a_2] \) and \( B' := P_1 \cup e' \cup A[a'_9, a'] \cup e'_9 \cup B[b'_9, b''_2] \cup B_0 \).

Now, since \( A, B \) is a good frame, then the existence of \( A', B', A[a_1, a''_9] \cup e''_9 \cup B[b''_9, b_2], \) and \( A[a_1, a_3] \cup e_3 \cup B[b_1, b_3] \) shows \( \alpha(A, B) = 2 \), a contradiction.

Thus, such \( P_1, P_2 \) do not exist. Then \( G_0 \) has a separation \( (H_1, H_2) \) with \( V(H_1 \cap H_2) = \{b'_1, b''_2\} \) such that \( b'_1 \in B(b_1, b_3), B[b^*_1, b'] \subseteq H_1, \) and \( \{a_0, b_1, b_2\} \subseteq H_2. \) We may assume \( b''_2 \in B[b'', b'_1] \) as otherwise we have (i).

Now, suppose \( G \) has no edge from \( B(b'_2, b'_1) \) to \( A \). then, combined with (2.1.2), \( G \) has a separation \( (K_1, K_2) \) of order 5, such that \( V(K_1 \cap K_2) = \{b_1, b'_1, b''_2, b'_1, a_0\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(K_1), \) and \( V(B[b_1, b'_1] \cup B[b''_2, b'_1]) \subseteq V(K_2) \), a contradiction.

So we may assume \( G \) has an edge \( e_0 \) from \( b_0 \in B(b'_2, b'_1) \) to \( a_0 \in A \). By the choice of \( e', e'', a_0 \in A[a_4, a''] \). So \( (e_3, e'', e_0, e_6, e_7) \) is a 5-edge configuration. Now, by Lemma 2.0.9 and 6.0.3, and by the existence of \( \{b'_1, b''_2\} \), (i) or (ii) holds, or it contradicts the choice of \( b'_1, b''_2 \).

**Subcase 2.2** \( a_8 \in A[a_1, a_3] \).

Note that if \( b_3 = b_4 \) we have symmetry between \( e_3 \) and \( e_4 \), so by Subcase 2.1, we may assume that

(2.2.1) if \( b_3 = b_4 \) then there exists \( e_9 = a_4b_9 \in E(G) \) with \( b_9 \in B(b_4, b'_1) \).

(2.2.2) The following holds: (a) \( G \) has no edge from \( B(b_4, b_7) \) to \( A(a_4, a_2) \) and so \( a_6 \notin A(a_4, a_2) \); and (b) \( G \) has no edge from \( B(b_3, b_7) \) to \( A[a_1, a_3] \) and so \( a_8 = a_3 \).

We have (a) to avoid a doublecross formed by such an edge and \( e_4, e_7, e_8 \). Now suppose (b) fails, and let \( e' \) be an edge from \( B(b_3, b_7) \) to \( A[a_1, a_3] \). If \( b_3 \neq b_4, e_3, e_4, e', e_7 \) form a doublecross. So \( b_3 = b_4 \). Then by (2.2.1), \( e_3, e_9, e', e_7 \) form a doublecross.
(2.2.3) There exists \( v \in \{a_4, b_4\} \) such that all edges from \( B[b_1, b'_1] \) to \( A(a_3, a_2) \) must contain \( v \).

Since we have proved \( a_8 \in A[a_1, a_3] \cup A(a_4, a_2) \), then, combined with (2.2.2), all edges from \( B(b_4, b'_4) \) to \( A \) must end in \( \{a_3, a_4\} \).

Now, we claim that \( G \) has no edge from \( B[b_1, b_4] \) to \( A(a_3, a_2) \). For, let \( e = ab \in E(G) \) with \( b \in B[b_1, b_4] \) and \( a \in A(a_3, a_2) \). Then \( a \in A(a_4, a_2) \), to avoid the doublecross \( e, e_4, e_5, e_8 \). So \( b = b_1 \) by the choice of \( \mathcal{P} \). Then \( a \neq a_2 \); so \( G \) has an edge \( e_2 = a_2 b' \) with \( b' \in B(b_1, b_2) \). Then \( b' \in B[b_7, b_2] \) to avoid the doublecross \( e_2, e_7, e_8, e' \). If \( b_3 \neq b_4 \) then \((e_2, e_7, e_4, e_3, e) \) contradicts the choice of \( \mathcal{P} \). So \( b_3 = b_4 \). Then \( e_9 \) is defined by (2.2.1). Hence, \((e_2, e_7, e_9, e_3, e) \) contradicts the choice of \( \mathcal{P} \).

Thus, if (2.2.3) fails, then there exist \( e' = a_4 b', e'' = a'' b_4 \) with \( a'' \in A(a_3, a_2) - a_4 \) and \( b' \in B(b_1, b'_1) \). By the choice of \( \mathcal{P} \), \( a'' \in A(a_3, a_4) \). So \( e_8, e', e'', e_7 \) form a doublecross, a contradiction.

(2.2.4) \( a_1 = a_3 \).

For, suppose \( a_1 \neq a_3 \). Then there exists \( e_1 = a_1 b \in E(G) \) with \( b \in B(b_1, b_2) \). Indeed, \( b \notin B(b_1, b_4) \) by the choice of \( \mathcal{P} \); \( b \notin B(b_4, b_7) \) by (2.2.2). So \( b \in B[b_7, b_2] \). Moreover, \( b_3 = b_4 \), for, otherwise, \((e_1, e_7, e_8, e_4, e_3) \) contradicts the choice of \( \mathcal{P} \). Thus \( e_9 \) in (2.2.1) is defined.

We claim that \( G \) has no edge from \( B[b_1', b_7] \) to \( A(a_1, a_7) \). For such an edge and \( e_1, e_7, e_9, e_3 \) form a 5-edge configuration. Hence, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of \( b_1', b_2' \).

Thus, combined with (2.2.2), \( a_6 \in A(a_7, a_4) \). So \((e_6, e_1, e_5, e_9, e_3) \) is a 5-edge configuration. Hence, \( b_2' = b_2 \); or else, by Lemma 2.0.9 and 6.0.3, (i) or (ii) holds, or it contradicts the choice of \( b_1', b_2' \).

Now, we claim that \( \{b_1, b_2, b_1', a_3, a_4\} \) is a cut in \( G \) separating \( \{a_1, a_2\} \) from \( \{a_0\} \), a contradiction to that \( G^* \) is 6-connected. In fact, since \( b_2' = b_2 \), we just need to show that \( G \)
has no edge from $B(b_1, b'_1)$ to $A[a_1, a_2] - \{a_3, a_4\}$. Suppose there exists $e^* = a^*b^* \in E(G)$ with $a^* \in A[a_1, a_2] - \{a_3, a_4\}$ and $b^* \in B(b_1, b'_1)$. By (2.2.3) and by the existence of $e_9, a^* \in A[a_1, a_3)$. Then $b^* \notin B(b_1, b'_1)$; otherwise, $(e^*, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then, $b^* \in B(b_3, b'_1)$, and $e^*, e_3, e_6, e_7$ form a doublecross. Hence, our claim is true, which finishes the proof of (2.2.4). □

Let $e_2 = a'_2b' \in E(G)$ with $a'_2 \in V(A)$ and $b' \in B(b'_1, b'_2)$, such that $A[a_2, a'_2]$ is minimal. We may assume that

\[(2.2.5) \text{ there exists } e_0 = a_0b_0 \in E(G) \text{ with } a_0 \in A(a_1, a'_2) \text{ and } b_0 \in B - B[b'_1, b'_2].\]

For otherwise, $G$ has a separation $(H_1, H_2)$ such that $V(H_1 \cap H_2) = \{b'_1, b'_2, a_1, a'_2\}$ with $a_0, a_1, a_2, b_1, b_2 \in V(H_1)$, and $b_6 \in V(H_2)$, a contradiction. □

\[(2.2.6) \text{ } b_0 \in B[b'_1, b'_2] \text{ for every choice of } e_0.\]

For, otherwise, $b_0 \in B(b'_2, b'_2)$. Then $a_0 \in A(a_1, a_2)$, to avoid the doublecross $e_4, e_8, e_2, e_0$. Also, $a_6 \in A[a_5, a_0]$; otherwise $(e_3, e_4, e_5, e_6, e_0)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_7 \in A[a_6, a_0]$; or else $(e_3, e_4, e_6, e_7, e_0)$ contradicts the choice of $\mathcal{P}$. But this shows that $a_6 \in A[a_5, a_7]$, a contradiction. □

Now, combined with (2.2.3), $G$ has a separation $(H_1, H_2)$ such that $V(H_1 \cap H_2) = \{a_1, a'_2, b'_1, b'_2, v\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a'_2] \cup B[b'_1, b'_2]) \subseteq V(H_2)$, a contradiction. □

**Lemma 6.0.6** Suppose $\gamma$ is infeasible, and $A', B'$ is a core $a_0$-frame in $\gamma$. Suppose $\mathcal{P} = (e_3, e_4, e_5, e_6, e_7)$ is an ideal 5-edge configuration w.r.t. an ideal frame $A, B$ in $\gamma$. Then (i)–(iv) of Lemma 6.0.5 do not hold.

**Proof.** For notation convenience, we assume $a_1, a_3, a_4, a_2$ occur on $A$ in order, and $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ occur on $B$ in order. And we choose $b'_1, b'_2$ satisfying the conclusions of Lemma
6.0.5 so that $b'_1, b'_2$ are defined from (i) or (ii) whenever possible, subject to this, $B[b_1, b'_1]$ is minimal, and subject to this, $B[b_7, b'_2]$ is minimal.

Moreover, we define $t_0, t_1, t_2 \in V(G), B(t_1, t_2), B[b_1, t_1], \text{ and } B(t_2, b_2)$ according to \{b'_1, b'_2\}.

(i) Suppose (i) of Lemma 6.0.5 occurs. We define $t_1 := b'_1, t_2 := b'_2, t_0 := a'_0, B(t_1, t_2) := B(b'_1, b'_2), B[b_1, t_1] := B[b_1, b'_1], \text{ and } B(t_2, b_2) := B(b'_2, b_2).

(ii) Suppose (ii) of Lemma 6.0.5 occurs. We define $t_0 := t_1 := b'_1, t_2 := b'_2, t_0 := a'_0, B(t_1, t_2) := B(b'_1, b'_2), B[b_1, t_1] := B[b_1, b'_1], \text{ and } B(t_2, b_2) := B(b'_2, b_2).

(iii) Suppose (iii) of Lemma 6.0.5 occurs.

(a) If $G$ has no edge from $B(b'_2, b_7)$ to $A$, we define $t_1 := b'_1, t_2 := b_7, t_0 := b'_2, B(t_1, t_2) := B(b'_1, b_7), B[b_1, t_1] := B[b_1, b'_1], \text{ and } B(t_2, b_2) := B(b_7, b_2).

(b) If $G$ has an edge $f_7$ from $b_7^* \in B(b'_2, b_7)$ to $a_7$, we define $t_1 := b'_1, t_2 := a_7, t_0 := b'_2, B(t_1, t_2) := B(b'_1, b'_2), B[b_1, t_1] := B[b_1, b'_1], \text{ and } B(t_2, b_2) := B[b_7, b_2].

(iv) Suppose (iv) of Lemma 6.0.5 occurs.

(a) If $G$ has no edge from $B(b_4, b'_1)$ to $A$, we define $t_1 := b_4, t_2 := b'_2, t_0 := b'_1, B(t_1, t_2) := B(b_4, b'_2), B[b_1, t_1] := B[b_1, b_4], \text{ and } B(t_2, b_2) := B(b'_2, b_2).

(b) If $G$ has an edge $f_4$ from $b_4^* \in B(b_4, b'_1)$ to $a_4$, we define $t_1 := a_4, t_2 := b'_2, t_0 := b'_1, B(t_1, t_2) := B(b'_1, b'_2), B[b_1, t_1] := B[b_1, b_4], \text{ and } B(t_2, b_2) := B(b'_2, b_2).

Note that by Lemma 6.0.5, when (iii) (b) occurs, $G$ has no edge from $B(b'_2, b_7)$ to $A[a_1, a_2] - a_7$; when (iv) (b) occurs, $G$ has no edge from $B(b_4, b'_1)$ to $A[a_1, a_2] - a_4$.

Let $a_1^*, a_2^*$ be vertices on $A$ such that $a_1, a_1^*, a_2^*, a_2$ occur on $A$ in that order, $G$ has edges $f_i, i = 1, 2, \text{ from } a_1^* \text{ to } b_i^* \in B(t_1, t_2)$, and subject to these, $A[a_1^*, a_2^*]$ is maximal. Notice that $A[a_5, a_6] \subseteq A[a_1^*, a_2^*].$

Case 1. $G$ has no edge from $B[b_1, t_1]$ to $A(a_1^*, a_2^*)$, which is different from $e_4$.  

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Now, as $G^*$ is 6-connected, $\{t_0, t_1, t_2, a'_1, a'_2\}$ is not a cut in $G$ separating $V(A(a'_1, a'_2) \cup B(t_1, t_2))$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Thus, combined with Lemma 6.0.5, $G$ has an edge $e_8$ from $b_8 \in B(t_2, b_2)$ to $a_8 \in A(a'_1, a'_2)$ (or $a_8 \in A(a'_1, a'_2) - a_7$ if (iii)(b) occurs). Then, obviously, $b_8 \in B(b'_2, b_2) \cap B[b_7, b_2]$.

We first see that $a_8 \in A(a_3, a_4)$. For, suppose $a_8 \notin A(a_3, a_4)$. Hence, $a_8 \in A(a_1, a_3) \cup A(a_4, a_2)$. If $a_8 \in A(a_1, a_3)$, then $a'_5 \in A(a_1, a_3)$, and so $e_3, f_1, e_5, e_8$ force a double-cross, or $(f_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Therefore, $a_8 \in A(a_4, a_2)$. Then $b'_2 \in B(b'_1, b_4)$; otherwise $e_4, e_5, f_2, e_8$ force a doublecross. But now, $(e_3, f_2, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

We claim that $b_8 = b_7$, and so (iii)(b) occurs with $a_8 \neq a_7$ and $f_7$ existing. For, suppose $b_8 \in B(b_7, b_2)$. Then $(e_3, e_4, e_5, e_6, e_8)$ (when $a_6 \notin A(a_5, a_8)$) or $(e_3, e_4, e_6, e_7, e_8)$ (when $a_6 \in A(a_5, a_8)$) contradicts the choice of $\mathcal{P}$.

Let $e = a_8b \in E(G)$ with $b \in B[b_1, b_2] - \{b_4, b_7\}$. We also claim that $b \in B[b_1, b_4]$.

First, by $b_8 = b_7$, $b \notin B(b_7, b_2)$. By (iii)(b), $b \notin B(b'_2, b_7)$. Moreover, $b \notin B(b_4, b'_2)$; otherwise, $(e_3, e_4, e, f_7, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. This finishes our claim.

Now, we may assume $a_8 \in A(a_3, a_7)$; otherwise, $e, e_4, f_7, e_8$ form a doublecross.

Then $a_7 \in A[a_1, a_5]$; otherwise, $(e_3, e_4, e_5, f_7, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

We see that $a_6 \in A(a_5, a_2)$. In fact, if $a_6 \in A(a_1, a_8)$, then $e_4, e_6, e_8, e$ form a doublecross; if $a_6 \in A(a_8, a_7)$, then $(e_3, e_4, e_6, f_7, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

$G$ has no edge from $B(b_4, b_5)$ to $A(a_1, a_5)$. For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A(a_1, a_5)$ and $b_9 \in B(b_4, b_5)$. Now, $a_9 \notin A[a_1, a_8]$ to avoid the doublecross $e, e_4, e_8, e_9$. Moreover, $a_9 \notin A[a_8, a_7]$ and $b_9 \notin B(b_4, b_5)$; or else, $(e_3, e_4, e_9, f_7, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. So $a_9 \in A[a_7, a_5]$ and $b_9 = b_5$, but then $(e_3, e_4, e_9, e_6, e_7)$ is a 5-edge configuration, contradicting the choice of $\mathcal{P}$.
\[ G \text{ has no edge from } B[b_7, b_2] \text{ to } A(a_5, a_2). \] For, otherwise, let \( e_9 = a_9b_9 \in E(G) \) with \( a_9 \in A(a_5, a_2) \) and \( b_9 \in B[b_7, b_2] \). Then \((e_9, e_8, f_7, e_6, e_5)\) forms a 5-edge configuration. Now, by Lemma 6.0.3, \( G_0 \) has a cut \( \{b''_1, b''_2\} \) or \( \{b''_1, b''_2, a''_0\} \) (w.r.t. \((e_9, e_8, f_7, e_6, e_5)\)) satisfying the conclusion of Lemma 6.0.3, such that \( b_1, b'_1, b'_2, b_2 \) occur on \( B \) in order. But then, by Lemma 2.0.9, \( G_0 \) has a cut, which contradicts the choice of \( \{b'_1, b'_2\} \).

Now, the existence of \( \{b'_1, b'_2\} \) contradicts Lemma 6.0.4.

**Case 2.** \( G \) has an edge \( e_8 \) from \( b_8 \in B[b_1, t_1] \) to \( a_8 \in A(a_1^*, a_2^*) \), such that \( e_8 \neq e_4 \).

Note that \( b_8 \in B[b_1, b_4] \cap B[b_1, b'_1] \). Now, we distinguish two subcases, \( a_8 \in A(a_5, a_2^*) \) and \( a_8 \in A(a_1^*, a_3] \).

**Subcase 2.1.** \( a_8 \in A(a_5, a_2^*) \).

We choose \( e_8 \) so that \( A[a_8, a_2] \) is maximal.

\((2.1.1)\) \( b_8 \notin B(b_1, b_4) \).

For, otherwise, \( b_8 \in B(b_1, b_4) \). Then \( a_8 \notin A(a_5, a_7) \) to avoid the doublecross \( e_8, e_4, e_5, e_7 \).

Now, \( b_3 = b_4 \) and \( a_8 \in A[a_1, a_4] \); otherwise, \((e_3, e_8, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \).

But then, \( e_3, e_4, e_7, e_8 \) form a doublecross. \( \square \)

\((2.1.2)\) \( b_8 \neq b_4 \).

For, otherwise, \( b_8 = b_4 \), and \((iv)(b)\) occurs with \( f_4 \) existing.

We see that \( a_8 \notin A(a_4, a_2) \); otherwise, \((e_3, e_8, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \).

So \( a_8 \in A(a_5, a_4) \).

\( G \) has no edge from \( A(a_5, a_4) \) to \( B(b_5, b_2) \); otherwise, such an edge together with \( e_5, e_8, f_4 \) forms a doublecross. Hence, \( a_7 \in A[a_1, a_5] \) and \( a_6 \notin A(a_5, a_4) \). Moreover, \( a_6 \notin A[a_1, a_7] \) to avoid the doublecross \( e_6, e_7, e_8, f_4 \). So \( a_6 \in A[a_4, a_2] \).

Now, since \( a_8 \) has degree at least 6 in \( G \), then \( G \) has an edge \( e_8' \) from \( a_8 \) to \( b_8' \in B[b_1, b_2] \setminus \{b_1, b_4, b_5\} \). Since \( G \) has no edge from \( A(a_5, a_4) \) to \( B(b_5, b_2) \), then, combined
with (2.1.1), \( b'_8 \in B(b_4, b_5) \). But then, \((e_3, e_4, e'_8, e_6, e_7)\) is a 5-edge configuration contradicting the choice of \( P \). □

Hence, \( b_8 = b_1 \) and \( b'_1 \neq b_1 \). Now, \( a_8 \in A[a_4, a'_8] \) to avoid the doublecross \( e_8, e_4, e_3, e_7 \). And \( b'_2 \in B(b_7, b'_2) \) to avoid the doublecross \( e_8, f_2, e_3, e_7 \). Moreover, \( b_3 = b_4 \); otherwise, \((f_2, e_7, e_4, e_3, e_8)\) contradicts the choice of \( P \).

Now, by no doublecross and by the choice of \( P \), \( G \) has no edge from \( B(b_1, b_3) \) to \( A \). Also, \( a_5 \in A[a_1, a_7] \), or else \((f_2, e_7, e_5, e_3, e_8)\) contradicts the choice of \( P \).

Finally, \( a_6 \in A[a_1, a_5] \), as, otherwise, \( e_8, e_6, e_3, e_7 \) (when \( a_6 \in A(a_8, a_2) \)) would form a doublecross, or \((f_2, e_7, e_6, e_3, e_8)\) (when \( a_6 \in A[a_5, a_8] \)) contradicts the choice of \( P \).

(2.1.3) \( G \) has no edge from \( B(b_6, b_2) \) to \( A[a_1, a_5] \) and no cross from \( B[b_6, b_2] \) to \( A[a_5, a_2] \).

For, let \( e = ab \in E(G) \) with \( b \in B(b_6, b_2) \) and \( a \in A[a_1, a_5] \). Then \( b = b_2 \); or else, \((e_3, e_4, e_5, e, e_7)\) (when \( b \notin B(b_6, b_7) \)) or \((f_2, e, e_5, e_3, e_8)\) (when \( b \in B[b_7, b_2] \)) contradicts the choice of \( P \). But then \( a \neq a_1 \), and \( e, e_8 \) and two edges from \( a_1, a_2 \) to \( B(b_1, b_2) \) would form a doublecross.

Moreover, suppose \( G \) has a cross from \( B[b_6, b_2] \) to \( A[a_5, a_2] \), then such a cross and \( e_5, e_6 \) would form a doublecross, a contradiction. □

(2.1.4) \( G \) has no edge from \( B(b_1, b_3) \) to \( A \).

For, otherwise, let \( e = ab \in E(G) \) with \( a \in A \) and \( b \in B(b_1, b_3) \). Then \( a \in A[a_4, a_8] \); or else, \((f_2, e_7, e_4, e, e_8)\) contradicts the choice of \( P \). But now, \((e, e_3, e_5, e_6, e_7)\) contradicts the choice of \( P \). □

(2.1.5) \( G \) has no edge from \( A(a_4, a_2) \) to \( B(b_1, b_7) \).

For, otherwise, let \( e = ab \in E(G) \) with \( a \in A(a_4, a_2) \) and \( b \in B(b_1, b_7) \). Then \( b \notin B(b_4, b_7) \) to avoid the doublecross \( e_4, e_6, e_7, e \). But then \( b \in B(b_1, b_4) \), and \((e, e_3, e_5, e_6, e_7)\) is a 5-edge configuration contradicting the choice of \( P \). □
Let $e^* = a_2b^* \in E(G)$, such that $b^* \in B(b_1,b_2)$, and $B[b^*,b_2]$ is minimal. Then by (2.1.3) and (2.1.5), $b^* \in B[b_7,b_2]$ and $G$ has no edge from $B(b^*,b_2)$ to $A$.

Let $e' = a'b' \in E(G)$ with $a' \in A(a_8,a_2)$ and $b' \in B(b_6,b_2)$, such that $B[b',b_2]$ is maximal. Note that $e'$ exists because of $e^*$. And $b' \in B[b_7,b^*]$ by (2.1.3).

Now, by (2.1.3), (2.1.5), and the choice of $e^*, e'$, we have

(2.1.6) $G$ has no edge from $B(b^*,b_2)$ to $A$ and no edge from $B(b_1,b')$ to $A(a_8,a_2)$.

(2.1.7) $G$ has no edge from $b_1$ to $A[a_1,a_8]$.

For, suppose there exists $e = ab_1 \in E(G)$ with $a \in A[a_1,a_8]$. Then, obviously, by the choice of $e_8, a \notin A(a_5,a_8)$. Hence, $a \in A[a_1,a_5]$. Since $a \neq a_1$, then let $e_0 = a_1b_0 \in E(G)$ with $b_0 \in B(b_1,b_2)$. Now, $b_0 \in B[b_7,b_2]$ to avoid the doublecross $e_0, e_4, e_7, e$. But then $(e_0, e^*, e_5, e_4, e)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. $\square$

(2.1.8) For any edge $f'_8 = a'_8b'_8 \in E(G)$ with $a'_8 \in A[a_5,a_2]$ and $b'_8 \in B(b_6,b_2)$, $G$ has no edge from $B(b_4,b'_8)$ to $A(a'_8,a_2)$.

For, otherwise, let $f'_9 = a'_9b'_9 \in E(G)$ with $a'_9 \in A(a'_5,a_2)$ and $b'_9 \in B(b_4,b'_8)$. Then $b'_9 \notin B(b_5,b'_8)$ to avoid the doublecross $e_5, e_6, f'_8, f'_9$. So $b'_9 \notin B(b_4,b_5)$. Moreover, $b'_9 \notin B(b_3,b_5)$; otherwise, $(e_3, e_4, f'_9, e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. So $b'_9 = b_5$. Now, we see that $a_7 \in A[a_5,a'_9]$; or else, $(e_3, e_4, f'_9, e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. But then $(e^*, e_7, f'_9, e_3, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. $\square$

(2.1.9) $G_0$ does not have a cut $\{b_3, b''\}$ or $\{b_3, b'', a''\}$ with $b'' \in B[b_6,b']$ separating $B[b_3,b'']$ from $\{a_0, b_1, b_2\}$.

For, suppose $G_0$ has such a cut $\{b_3, b''\}$ or $\{b_3, b'', a''\}$ with $b'' \in B[b_6,b']$. Now, let $a'_0 \in A[a_1,a_2]$ such that $G$ has an edge $f'_9$ from $a'_9$ to $b'_9 \in B(b_3,b'')$, and subject to this, $A[a'_0,a_2]$ is minimal. Obviously, by the existence of $e_5, a_9 \in A[a_5,a_2]$.

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We claim that \( a'_9 \notin A(a_8, a_2) \), and so by (2.1.7), \( G \) has no edge from \( b_1 \) to \( A[a_1, a'_9] \). For otherwise, \( b'_9 \notin B(b_3, b_7) \) to avoid the doublecross \( e_6, e_7, e_8, f'_9 \). But then \( b'_9 \in B[b_7, b'_9] \), and \( f'_9 \) contradicts the choice of \( e' \).

By (2.1.3) and (2.1.8), \( G \) has no edge from \( B(b'', b_2) \) to \( A[a_1, a_9] \). Thus, \( \{ a_1, b_3, b'', a_9 \} \) or \( \{ a_1, b_3, b'', a'', a_9 \} \) is a cut in \( G \) separating \( V(A[a_1, a_9] \cup B[b_3, b'']) \) from \( \{ a_0, a_1, a_2, b_1, b_2 \} \), a contradiction. \( \square \)

Since \( (e', e_6, e_5, e_3, e_8) \) is a 5-edge configuration, \( G_0 \) has a cut \( \{ b''_1, b''_2 \} \) or \( \{ b'_1, b'_2, a''_9 \} \) (w.r.t. \( (e', e_6, e_5, e_3, e_8) \)) satisfying the conclusion of Lemma 6.0.3, such that \( b_1, b'_1, b''_2, b_2 \) occur on \( B \) in order.

Moreover, since \( (e_3, e_4, e_5, e_6, e_7) \) is a 5-edge configuration, \( G_0 \) has a cut \( \{ b'^1_1, b'^1_2 \} \) or \( \{ b'^1_1, b'^1_2, a''_0 \} \) (w.r.t. \( (e_3, e_4, e_5, e_6, e_7) \)) satisfying the conclusion of Lemma 6.0.3, such that \( b_1, b'^1_1, b''_2, b_2 \) occur on \( B \) in order.

**Subcase 2.1.a.** Conclusions (i), or (ii), or (iii) of Lemma 6.0.3 holds for \( \{ b''_1, b''_2 \} \) or \( \{ b'_1, b'_2, a''_9 \} \) w.r.t. \( (e', e_6, e_5, e_3, e_8) \).

We may assume conclusion (iv) of Lemma 6.0.3 holds for \( \{ b'^1_1, b'^1_2 \} \) w.r.t. \( (e_3, e_4, e_5, e_6, e_7) \), and so \( b'^1_1 \in B(b_4, b_5) \) and \( b'^1_2 \in B(b_7, b_2) \). For otherwise, assume conclusions (i), or (ii), or (iii) of Lemma 6.0.3 holds for \( \{ b'^1_1, b'^1_2 \} \) or \( \{ b'^1_1, b'^1_2, a''_0 \} \) w.r.t. \( (e_3, e_4, e_5, e_6, e_7) \). Then by the choice of \( \{ b'_1, b'_2 \} \) with \( b'_1 \neq b_1 \), and by Lemma 2.0.8 and 2.0.9, we could find a cut \( \{ b_3, b'' \} \) or \( \{ b_3, b'', a'' \} \) with \( b'' \in B[b_6, b'] \) in \( G_0 \), which separates \( B[b_3, b''] \) from \( \{ a_0, b_1, b_2 \} \), a contradiction to (2.1.9).

Now, suppose conclusion (i) of Lemma 6.0.3 holds for \( \{ b'_1, b'_2, a''_9 \} \) w.r.t. \( (e', e_6, e_5, e_3, e_8) \). Then \( b'_2 \in B[b_6, b'_7] \) by \( b'_1 \neq b_1 \) and the choice of \( \{ b'_1, b'_2 \} \). Moreover, by Lemma 2.0.9, \( b''_2 = b_2 \), and \( b'^1_1, b''_2, b_2, a_0 \) are on a common finite face of \( G_0 \). Let \( a'_8 \in A[a_1, a_2] \) such that \( G \) has an edge \( f'_8 \) from \( B(b''_2, b_2) \) to \( a'_8 \), and \( A[a'_8, a_2] \) is maximal. Now, by (2.1.3), (2.1.4), and (2.1.8), \( G \) has a separation \( (H_1, H_2) \), such that \( V(H_1 \cap H_2) = \{ b_1, b_2, b_4, b''_2, a'_8 \} \), \( \{ a_0, a_1, b_1, b_2 \} \subseteq V(H_1) \), and \( V(A[a'_8, a_2] \cup B[b''_2, b_2]) \subseteq V(H_2) \), a contradiction.

Suppose conclusion (ii) of Lemma 6.0.3 holds for \( \{ b'_1, b'_2 \} \) w.r.t. \( (e', e_6, e_5, e_3, e_8) \). So
\( b_1'' = b_1 \) and \( b_2'' \in B[b_0, b'] \). Then by Lemma 2.0.9, \( \{b_1, b_2''\} \) is a cut in \( G_0 \) separating \( B[b_1, b_2''\} from \{b_1, b_2, a_0\} \), which contradicts the choice of \( \{b_1', b_2'\} \) (by \( b_1' \neq b_1 \)).

So we may assume conclusion (iii) of Lemma 6.0.3 holds for \( \{b_1'', b_2''\} \) w.r.t. \( (e', e_6, e_5, e_3, e_8) \).

Now, \( b_1'' \in B(b_1, b_3) \) and \( b_2'' \in B[b_0, b'] \). Then by Lemma 2.0.9, \( \{b_1'', b_2''\} \) is a cut in \( G_0 \) separating \( B[b_1'', b_2''] \) from \( \{b_1, b_2, a_0\} \). Now, let \( a'_9 \in A[a_4, a_2] \), such that \( G \) has an edge \( f'_9 \) from \( a'_9 \) to \( b_9 \in B[b_4, b_2''] \), and subject to this, \( A[a'_9, a_2] \) is minimal.

We see that \( G \) has no edge from \( B(b_2'', b_2) \) to \( A[a_1, a'_9] \). For, otherwise, let \( f'_8 = a'_8b_8 \in E(G) \) with \( a'_8 \in A[a_1, a'_9] \) and \( b_8 \in B(b_2'', b_2) \). Since \( B(b_2'', b_2) \subseteq B(b_7, b_2) \), then \( a'_8 \notin A[a_5, a_4] \); or else, \( (e_3, e_4, e_5, e_6, e'_8) \) is a 5-edge configuration contradicting the choice of \( \mathcal{P} \).

By (2.1.3), \( a'_8 \in A[a_4, a_2] \). Now, by (2.1.8), \( b'_9 = b_4 \). But now, \( (e_3, f'_9, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

Thus, by (2.1.4), \( G \) has a separation \( (H_1, H_2) \), such that \( V(H_1 \cap H_2) = \{b_1, b_1'', b_2'', a'_9\}, \)
\( \{a_0, a_2, b_1, b_2\} \subseteq V(H_1) \), and \( V(A[a_1, a'_9] \cup B[b_1'', b_2'']) \subseteq V(H_2) \), a contradiction.

Subcase 2.1.b. Conclusion (iv) of Lemma 6.0.3 holds for \( \{b_1'', b_2''\} \) w.r.t. \( (e', e_6, e_5, e_3, e_8) \).

Then \( b_2'' \in B[b_5, b_0] \), \( b_1'' = b_1 \), and \( \{b_1, b_2''\} \) is a cut in \( G_0 \) separating \( B[b_1, b_2''] \) from \( \{a_0, b_1, b_2\} \). By Lemma 2.0.9, the choice of \( \{b_1', b_2'\} \), and \( b_1' \neq b_1 \), we have \( b_2' = b_2, b_1' \in B(b_1, b_3) \), and \( b_1', b_2' \) are cut vertices of \( G_0 \) separating \( b_1 \) from \( \{a_0, b_2\} \). So \( \alpha(A, B) \leq 1 \).

We may assume \( b^* = b_7 \). For, suppose \( b^* \neq b_7 \), then \( b^* \in B(b_7, b_2) \). We first see that there does not exist a vertex \( u \in B[b^*, b_2] \), such that \( b_2', u \) are incident with a common finite face of \( G_0 \); or else, \( \{b_1'', b_2', u\} \) is a 3-cut in \( G_0 \) separating \( B[b_1'', u] \) from \( \{a_0, b_1, b_2\} \), a contradiction to the choice of \( \{b_1', b_2'\} \). Then we claim that \( G_0 - B[b_1, b_2''] - B[b^*, b_2] \) has disjoint paths \( B_2, A_0 \) from \( b_2, a_0 \) to \( b_7, b_0 \), respectively. For otherwise, since Subcase 2.1.a. does not hold, then, combined with the planar structure of \( G_0 \) and the choice of \( \{b_1', b_2'\} \), there exist \( u_0 \in V(G_0), u_2 \in B[b^*, b_2], \) and a separation \( (H_1, H_2) \) in \( G_0 \), such that \( V(H_1 \cap H_2) = \{b_1', u_0, u_2\}, V(B[b_1', b_2'') \cup B(b_2', u_2) \subseteq V(H_1 - H_2) \), and \( \{a_0, b_2\} \subseteq V(H_2 - H_1) \).

By (2.1.6), \( \{b_2'', u_0, u_2\} \) is a cut in \( G \) separating \( \{a_0, b_2\} \) from \( \{a_1, a_2, b_1\} \), a contradiction. Hence, \( B_2, A_0 \) exist. Now, let \( A' := A[a_1, a_6] \cup e_6 \cup A_0 \) and \( B' := B[b_1, b_3] \cup e_5 \cup A[a_5, a_7] \cup \)
$e_7 \cup B_2$. Then the existence of $A', B', e_8 \cup A[a_8, a_2]$, and $e^* \cup B[b', b_2]$ implies $\alpha(A, B) = 2$ (by Lemma 3.0.1), a contradiction.

Now, by (2.1.5) and (2.1.6), $G$ has a separation $(H_1, H_2)$, such that $V(H_1 \cap H_2) = \{b_1, b_7, a_4\}, \{a_0, a_1, b_1, b_2\} \subseteq V(H_1)$, and $V(A(a_4, a_2)) \subseteq V(H_2 - H_1)$, a contradiction.

Subcase 2.2. $a_8 \in A(a^*_1, a_5)$.

Now, we may assume $G$ has no edge from $B[b_1, t_1]$ to $A(a_5, a^*_2)$. Choose $e_8$ so that $A[a_8, a_5]$ is minimal, and subject to this, $B[b_8, b'_1]$ is minimal. Then $G$ has no edge from $B[b_1, b_4] \cap B[b_1, b'_1]$ to $A(a_8, a^*_2)$.

(2.2.1) $G$ has no cross from $B[b_1, b_4]$ to $A[a_1, a_5]$, and so $b_8 \in B[b_3, b_4]$. □

For, suppose $G$ has a cross from $B[b_1, b_4]$ to $A[a_1, a_5]$. Then such a cross together with $e_4, e_5$ forms a doublecross, a contradiction. Now, by the choice of $e_8$, we may assume $b_8 \in B[b_3, b_4]$. □

(2.2.2) $G$ has no edge from $B(b_8, b_7)$ to $A[a_1, a_8) \cap A[a_1, a_7)$, and so if $a_8 \in A[a_1, a_7]$, then $b^*_1 \in B[b_7, b_2]$.

For otherwise, such an edge together with $e_4, e_7, e_8$ (when $b_8 \neq b_4$) or $f_4, e_7, e_8$ (when $(iv)(b)$ occurs with $b_8 = b_4$) forms a doublecross, a contradiction. □

(2.2.3) $a_7 \in A[a_1, a_5]$.

For, suppose $a_7 \in A(a_5, a_2)$. Then $b^*_1 \in B[b_7, b_2]$ by (2.2.2). So $b_7 \neq b_2$ (by $b^*_1 \neq b_2$). Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, $b_8 \in B[b_1, b_4]$ and $(f_1, e_7, e_5, e_4, e_8)$ contradicts the choice of $P$. But then $(f_1, e_7, e_5, f_4, e_8)$ is a 5-edge configuration, so by Lemma 2.0.9 and 6.0.3, $G_0$ has a cut contradicting the choice of $\{b'_1, b'_2\}$. □

(2.2.4) $G$ has no edge from $B(b_5, b_7)$ to $A[a_1, a_7)$, and so $a_6 \in A(a_5, a_2)$.
For, otherwise, let $e_9 = a_9 b_9 \in E(G)$ with $a_9 \in A(a_1, a_7)$ and $b_9 \in B(b_5, b_7)$. Then $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2]$ by (2.2.2). So $b_7 \neq b_2$ (by $b_1^* \neq b_2$). Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. But then, $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration, so by Lemma 2.0.9 and 6.0.3, $G_0$ has a cut contradicting the choice of \{b'_1, b'_2\}. □

(2.2.5) $G$ has no edge from $B(b_4, b_5)$ to $A(a_1, a_5)$.

For, otherwise, let $e_9 = a_9 b_9 \in E(G)$ with $a_9 \in A(a_1, a_5)$ and $b_9 \in B(b_4, b_5)$. Then $a_9 \not\in A[a_7, a_5]$; otherwise, $(e_3, e_4, f_9, e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. Moreover, $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2]$ by (2.2.2). So $b_7 \neq b_2$ (by $b_1^* \neq b_2$).

Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. But then, $b_9 = b_5$, and $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration, so by Lemma 2.0.9 and 6.0.3, $G_0$ has a cut contradicting the choice of \{b'_1, b'_2\}. □

(2.2.6) $G$ has no edge from $B(b_6, b_2)$ to $A(a_5, a_2)$.

For, otherwise, let $e_9 = a_9 b_9 \in E(G)$ with $a_9 \in A(a_5, a_2)$ and $b_9 \in B(b_6, b_2)$. Then $b_9 \in B[b_4, b_2]$; or else, $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$.

We see that $b_9 \neq b_2$. For otherwise, $a_9 \neq a_2$. Let $e = a_9 b \in E(G)$ with $b \in B(b_1, b_2)$ and $b \neq b_4$. Then $b \not\in B(b_1, b_4)$; otherwise, $(e_3, e_5, f_1, e_9)$ contradicts the choice of $\mathcal{P}$.

So $b \in B(b_4, b_2)$. Now, $e_8, e_9, f_1, e$ form a doublecross, a contradiction.

So $b_9 \in B[b_7, b_2]$ and $b_7 \neq b_2$. Now, we may assume $(iv)(b)$ occurs with $b_8 = b_4$; otherwise, combined with (2.2.2), $(e_9, e_7, e_5, e_4, e_8)$ (when $a_7 \in A[a_1, a_7]$) or $(e_9, f_1, e_5, e_4, e_8)$ (when $a_8 \in A[a_1, a_7]$) contradicts the choice of $\mathcal{P}$. But then, $(e_9, e_7, e_5, f_4, e_8)$ (when $a_7 \in A[a_1, a_8]$) or $(e_9, f_1, e_5, f_4, e_8)$ (when $a_8 \in A[a_1, a_7]$) is a 5-edge configuration. By Lemma 2.0.9 and 6.0.3, $G_0$ has a cut contradicting the choice of \{b'_1, b'_2\}. □

Now, by (2.2.3)–(2.2.6) and by Lemma 6.0.4,

(2.2.7) $G_0$ does not contain a cut $\{b'_1, b'_2\}$ separating $B[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ with $b'_1 \in B[b_1, b_4]$ and $b'_2 \in B[b_6, b_2]$.
By (2.2.7), we have

\begin{equation}
(2.2.8) \text{Conclusions (ii) and (iii) of Lemma 6.0.5 do not hold for } \{b'_1, b'_2\} \text{ w.r.t. } (e_3, e_4, e_5, e_6, e_7). \nonumber
\end{equation}

(2.2.9) \text{G has no edge from } B[b_1, b_4] \text{ to } A(a_5, a_2). \nonumber

For, suppose \(G\) has an edge \(e\) from \(b \in B[b_1, b_4]\) to \(a \in A(a_5, a_2)\). Then we may assume \(b = b_1\). For otherwise, \(b \in B(b_1, b_4)\). Now, \(a \in A(a_5, a_4)\) and \(b \in B(b, b_4)\); or else, \((e_3, e, e_5, e_6, e_7)\) contradicts the choice of \(\mathcal{P}\). But then \(e_3, e_4, e, e_5\) form a doublecross.

So \(a \neq a_2\) (by \(b = b_1\)). Now, let \(e_0 = a_2b_0 \in E(G)\) with \(b_0 \in B(b_1, b_2)\). If \(b_0 \in B(b_1, b_7)\), then \(e_0, e, e_3, e_7\) form a doublecross. So, \(b_0 \in B(b_7, b_2)\), contradicting (2.2.6). \qed

\begin{equation}
(2.2.10) \text{G has no parallel edges from } A[a_1, a_8] \text{ to } B[b_1, b_2] \text{ and no parallel edges from } A[a_1, a_5] \text{ to } B[b_6, b_2]. \nonumber
\end{equation}

For otherwise, such two parallel edges together with \(e_4, e_8\) or \(e_5, e_6\) form a doublecross, a contradiction. \qed

Let \(e'_7 = a'_7b'_7 \in E(G)\) with \(a'_7 \in A(a_1, a_7)\) and \(b'_7 \in B(b_7, b_2)\), such that \(A[a_1, a'_7] \cup B[b'_7, b_2]\) is minimal.

\begin{equation}
(2.2.11) a'_7 \in A[a_1, a_8], \text{ and } G \text{ has no edge from } B(b'_7, b_2) \text{ to } A. \nonumber
\end{equation}

For, suppose \(a'_7 \notin A[a_1, a_8]\). Since \(a'_7 \in A[a_1, a_8]\), then by the choice of \(e'_7, b'_1 \in B(b_8, b'_7)\), and so \(e_8, e_4, f_1, e'_7\) form a doublecross.

By (2.2.6) and (2.2.10), and by the choice of \(e'_7\), \(G\) has no edge from \(B(b'_7, b_2)\) to \(A\). \qed

Let \(e' = a'b' \in E(G)\) with \(a' \in A[a_1, a_5]\) and \(b' \in B[b_1, t_1]\), such that \(A[a_1, a'] \cup B[b_1, b']\) is minimal.

By (2.2.1) and (2.2.9), and by the choice of \(e'\), we have

\begin{equation}
(2.2.12) e', e_8 \text{ do not form a cross, and } G \text{ has no edge from } B(b'_1, b'_8) \text{ to } A, \text{ and no edge from } B(b'_8, b_1) \text{ to } A[a_1, a'] \cup A(a_8, a_2). \nonumber
\end{equation}
(2.2.13) If conclusion \((iv)\) of Lemma 6.0.5 holds for \(\{b'_1, b'_2\}\) w.r.t. \((e_3, e_4, e_5, e_6, e_7)\), then there does not exist a 3-cut \(\{b''_1, b''_2, a''_0\}\) in \(G_0\) with \(b''_1 \in B[b_1, b_4]\) and \(b''_2 \in B(b_5, b_2)\), which separates \(B[b''_1, b''_2]\) from \(\{a_0, b_1, b_2\}\).

For, suppose conclusion \((iv)\) of Lemma 6.0.5 holds for \(\{b'_1, b'_2\}\) w.r.t. \((e_3, e_4, e_5, e_6, e_7)\), and (2.2.13) fails. Then \(b'_1 \in B(b_4, b_5)\), \(b'_2 \in B(b_7, b_2)\), and \(G\) has no edge from \(B(b_4, b'_1)\) to \(A - a_4\). Now, by the choice of \(\{b'_1, b'_2\}\) and by Lemma 2.0.9, \(b'_1 = b_1, b'_2 \in B(b_5, b_7)\), \(a''_0 = a_0, b'_2 = b_2\), and \(\alpha(A, B) \leq 1\).

By the choice of \(\{b'_1, b'_2\}\), and by planar structure of \(G_0\), we may assume \(G_0 - a_0 - B[b'_7, b_2]\) contains a path \(B_2\) from \(b_2\) to \(b'_2\).

Let \(e'_4 = a_4b'_4 \in E(G)\) with \(b'_4 \in B[b_4, b'_1]\) such that \(B[b'_4, b'_1]\) is minimal. Since \(b_8 \in B[b_1, t_1]\), then \(b_8 \neq b'_4\).

We claim that if \(b'_4 \neq b_4\), then \(G\) has no edge from \(B[b_1, b'_4]\) to \(A(a_5, a_2) - a_4\). For, suppose \(b'_4 \in B(b_4, b'_1)\) and \(G\) has an edge \(e\) from \(b \in B[b_1, b'_4]\) to \(a \in A(a_5, a_2) - a_4\). Now, by (2.2.9), \(b \neq B[b_4, b'_4]\). By \((iv)\) of Lemma 6.0.5, \(b \notin B[b_4, b'_4]\). So \(b = b_4\). By the choice of \(P\), \(a \in A(a_5, a_4)\). Now, let \(e_0 = ab_0 \in E(G)\) with \(b_0 \in B[b_1, b_2]\) and \(b_0 \notin \{b_1, b_5\}\). By (2.2.9), \(b_0 \notin B[b_1, b_4]\). Moreover, \(b_0 \notin B[b_5, b_2]\) to avoid the doublecross \(e, e_0, e'_4, e_5\). So \(b_0 \in B[b_4, b_5]\). But then \(e, e_6, e'_4, e_5\) form a doublecross (when \(a_6 \in A(a_5, a_4)\), or \((e_3, e_4, e_0, e_6, e_7)\) contradicts the choice of \(P\) (when \(a_6 \in A[a_4, a_2]\)).

By the choice of \(e_8\), (2.2.1), (2.2.9), \((iv)\) of Lemma 6.0.5, and our previous claim, if \(b'_4 = b_4\), then \(G\) has no edge from \(B(b_8, b'_4)\) to \(A\); if \(b'_4 \neq b_4\), then \(G\) has no edge from \(B(b_8, b'_4)\) to \(A - a_4\).

Now, we see that \(e' \cap e_8 = \emptyset\). For, suppose there exists a vertex \(v \in e' \cap e_8\). Then, by (2.2.12), \((iv)\) of Lemma 6.0.5, and our previous analysis, \(\{b_1, v, b_4, b'_1, b_2\}\) (when \(b'_4 = b_4\)) or \(\{b_1, v, a_4, b'_1, b_2\}\) (when \(b'_4 \neq b_4\)) is a cut in \(G\) separating \(a_0\) from \(V(A)\), a contradiction.

We may assume \(G_0 - B(b_1, b'_1) - B[b'_1, b_2]\) contains disjoint paths \(B_1, A_0\) from \(b_1, a_0\) to \(b_8, b'_4\), respectively. For, suppose not. There exists a vertex \(v \in V(G_0)\), such that \(v\) is a cut vertex in \(G_0 - B(b_1, b'_1) - B[b'_1, b_2]\) separating \(b_1, a_0\) from \(b_8, b'_4\). We see that \(v \notin B[b'_1, b_8]\);
otherwise, \(v\) and \(b'_1\) are incident with a common finite face of \(G_0\), and so \(\{v, b'_1, b'_2\}\) is a 3-cut in \(G_0\) separating \(B[v, b'_2]\) from \(\{a_0, b_1, b_2\}\), a contradiction to the choice of \(\{b'_1, b'_2\}\).

Moreover, \(v \notin B[b'_4, b'_1]\). For otherwise, there exists a vertex \(v_1 \in B(b_1, b')\), such that \(v_1, v\) are incident with a common finite face of \(G_0\). By (2.2.12), (iv) of Lemma 6.0.5, and the choice of \(e'_4\), \(\{v_1, v, b'_1\}\) is a cut in \(G\) separating \(\{a_0, b_1\}\) from \(\{a_1, a_2, b_2\}\), a contradiction.

Now, we may assume \(v \notin V(B)\), and so there exists a vertex \(v_1 \in B(b_1, b')\), such that \(v_1, v\) are incident with a common finite face of \(G_0\), and \(v, b'_1\) are incident with a common finite face of \(G_0\). But then, combined with (2.2.12), \(\{v, v, b'_1\}\) is still a cut in \(G\) separating \(\{a_0, b_1\}\) from \(\{a_1, a_2, b_2\}\), a contradiction.

Now, combined with Lemma 3.0.1, the path \(B_1 \cup e_8 \cup A[a_8, a_5] \cup e_5 \cup B[b_5, b'_2] \cup B_2\) from \(b_1\) to \(b_2\), the path \(A[a_4, a_2] \cup e'_4 \cup A_0\) from \(a_2\) to \(a_0\), the path \(A[a_1, a'_1] \cup e'_7 \cup B[b'_7, b_2]\) from \(a_1\) to \(b_1\), and the path \(A[a_1, a'_1] \cup e'_7 \cup B[b'_7, b_2]\) from \(a_1\) to \(b_2\) show that \(\alpha(A, B) = 2\), a contradiction. □

(2.2.14) Conclusion (i) of Lemma 6.0.5 holds for \(\{b'_1, b'_2\}\) w.r.t. \((e_3, e_4, e_5, e_6, e_7), b_8 \neq b_4\), and \(G\) has no edge from \(B[b_1, b'_1]\) to \(A(a_8, a_2)\).

For, suppose conclusion (i) of Lemma 6.0.5 does not hold for \(\{b'_1, b'_2\}\). By (2.2.8), conclusion (iv) of Lemma 6.0.5 holds for \(\{b'_2, b'_2\}\) w.r.t. \((e_3, e_4, e_5, e_6, e_7)\). So \(b'_1 \in B(b_4, b_5)\) and \(b'_2 \in B[b_7, b_2]\). By (2.2.1) and (2.2.5), \(b'_1 \in B(b_5, b_2)\). Hence, \((f_1, e_6, e_5, e_4, e_8)\) (when (iv)(a) occurs) or \((f_1, e_6, e_5, e_4, e_8)\) (when (iv)(b) occurs) is a 5-edge configuration. However, by Lemma 2.0.9 and 6.0.3, \(G_0\) has a cut contradicting (2.2.13) or the choice of \(\{b'_1, b'_2\}\).

Hence, conclusion (i) of Lemma 6.0.5 holds for \(\{b'_1, b'_2\}\) w.r.t. \((e_3, e_4, e_5, e_6, e_7)\), which implies that \(b'_1 \in B[b_1, b_4]\). Since \(b_8 \in B[b_1, b'_1]\), then \(b_8 \neq b_4\). By (2.2.9), \(G\) has no edge from \(B[b_1, b'_1]\) to \(A(a_5, a_2)\). Now, by the choice of \(e_8\), \(G\) has no edge from \(B[b_1, b'_1]\) to \(A(a_8, a_2)\). □

(2.2.15) \(G\) has no edge from \(B(b_8, b_6)\) to \(A(a_1, a_8)\), and so \((f_1, e_6, e_5, e_4, e_8)\) is a 5-edge configuration with \(b'_1 \in B[b_6, b_2]\).
We first claim that $G$ has no edge from $B(b_5, b_6)$ to $A(a_1, a_8)$. For, suppose there exists $e = ab \in E(G)$ with $b \in B(b_5, b_6)$ and $a \in A(a_1, a_8)$. We may assume $a_7 \in A(a_1, a)$ to avoid the doublecross $e_4, e_7, e_8, e$. But now, since $a_3 \in A(a_1, a_7)$, then, combined with (2.2.1), $b_3 \in B(b_1, b_8)$, and so $(e_3, e_8, e, e_7)$ contradicts the choice of $P$.

Now, by our claim, $b_1^* \notin B(b_5, b_6)$, and so $(f_1, e_6, e_5, e_4, e_8)$ is a $5$-edge configuration with $b_1^* \in B(b_6, b_2)$. \(\square\)

We choose $f_1$ so that $B[b_6, b_1^*]$ is minimal. Moreover, we let $e'_5 = a'_5b'_5 \in E(G)$ with $a'_5 \in A(a_1^*, a_6)$ and $b'_5 \in B[b_5, b_6]$ so that $B[b'_5, b_6]$ is minimal. Now, since $(f_1, e_6, e'_5, e_4, e_8)$ is a $5$-edge configuration, $G_0$ has a cut $\{b_1^#, b_2^#\}$ or $\{b_1^#, b_2^#, a_0^#\}$ (w.r.t. $(f_1, e_5, e'_5, e_4, e_8)$) satisfying the conclusion of Lemma 6.0.3, such that $b_1, b_1^#, b_2^#, b_2$ occur on $B$ in order.

By (2.2.7), we have

(2.2.16) Conclusions $(ii)$ and $(iii)$ of Lemma 6.0.3 do not hold for $\{b_1^#, b_2^#\}$ w.r.t. $(f_1, e_6, e'_5, e_4, e_8)$.

(2.2.17) Conclusion $(i)$ of Lemma 6.0.3 holds for $\{a_0^#, b_1^#, b_2^#\}$ w.r.t. $(f_1, e_6, e'_5, e_4, e_8)$.

For, otherwise, by (2.2.16), conclusion $(iv)$ of Lemma 6.0.3 holds for $\{b_1^#, b_2^#\}$ w.r.t. $(f_1, e_6, e'_5, e_4, e_8)$. So $b_1^# \in B[b_1, b_8]$ and $b_2^# \in B[b'_5, b_6)$. Then by Lemma 2.0.9, and by the choice of $\{b_1', b_2', b_2'\}$, we have $b_1^# = b_1, b_2 = b_2, a_0 = a'_0$, and $\alpha(A, B) \leq 1$. We further choose $\{b_1^#, b_2^#\}$ so that $B[b_2^#, b_2]$ is minimal.

By the choice of $\{b_1', b_2'\}$, and by planar structure of $G_0$, we may assume $G_0 - a_0 - B(b_1, b_1')$ contains a path $B_1$ from $b_1$ to $b_1'$.

We let $e'_6 = a'_6b'_6 \in E(G)$ with $a'_6 \in A(a_5, a_2)$ and $b'_6 \in B(b_2^#, b_6)$, such that $A[b'_6, b_2]$ is maximal.

$G$ has no edge from $B(b_5', b_6')$ to $A$. For, suppose $G$ has an edge $e$ from $b \in B(b_5', b_6')$ to $a \in A$. Then by the choice of $e'_5, a \in A[a_1, a_5]$. By the choice of $e'_5, a \notin A(a_1^*, a_6)$. So $a \in A[a_1, a_1^*]$, which contradicts (2.2.15).

Let $A_0$ be the path from $a_0$ to $b_6'$ on the boundary of $G_0 - B[b_1, b_2^#]$ without going through $b_2$. Since (2.2.17) fails, then combined with the choice of $\{b_1^#, b_2^#\}$, we may assume
We claim that $G$ has an edge $e'_7$ from $a'_7 \in A(a'_7, a_8)$ to $b'_7 \in B(b_6, b'_7)$. In fact, we see that $G$ has an edge $e$ from $a \in A(a_1, a_8)$ to $b \in B(b'_1, b_2) - \{b_6\}$, such that $e \cap e'_7 = \emptyset$; otherwise, by (2.2.1) and (2.2.10), $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{b_1, b'_1, a_8, b_6, u, a_1\}$ with $u \in \{a'_7, b'_7\}$, $V(A[a_1, a_8] \cup B[b_1, b'_1]) \subseteq V(G_1)$, and so $G$ is planar, a contradiction to Lemma 2.0.3.

Now, (2.2.15), $b \notin B(b_8, b_6)$, and so $b \in B(b_6, b_2)$. Finally, by (2.2.10) and the choice of $e'_7$, $a \in A(a'_7, a_8)$ and $b \in B(b_6, b'_7)$, which finishes our claim.

We further choose $e'_7$ with $e'_7 \in A(a'_7, a_8)$ and $b'_7 \in B(b_6, b'_7)$ so that $A[a_1, a'_7]$ is maximal. Now, we may assume $a'_7 \in A(a', a_8)$. For otherwise, $a'_7 \in A[a_1, a']$. By (2.2.10), (2.2.15), and the choice of $e'_7$, $\{b_1, b'_1, a'_7, a_8, b_6\}$ is a cut in $G$ separating $V(A[a', a_8] \cup B[b_1, b'_1])$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

We may also assume $G_0 - A_0 - B[b'_7, b_2]$ contains a path $B_2$ from $b_2$ to $b'_7$. For otherwise, $b'_7 \neq b_2$, and there exist a vertex $v_1 \in A_0$ and a vertex $v_2 \in B[b'_7, b_2]$, such that $v_1, v_2$ are incident with a common finite face in $G_0$. If $v_1 = a_0$, then $\{v_1, v_2, b_2\}$ is a cut in $G$ separating $N_G(b_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction; if $v_1 \neq a_0$, then combined with (2.2.11), $\{b_1, b_2', v_1, v_2, b_2\}$ is a cut in $G$ separating $a_0$ from $\{a_1, a_2\}$, a contradiction.

Now, combined with Lemma 3.0.1, the path $B_1 \cup B[b'_1, b_5] \cup e_5 \cup A[a'_7, a_8] \cup e'_7 \cup B_2$ from $b_1$ to $b_2$, the path $A[a_0', a_2] \cup e'_6 \cup A_0$ from $a_2$ to $a_0$, the path $A[a_1, a'] \cup e' \cup B[b_1, b'_7]$ from $a_1$ to $b_1$, and the path $A[a_1, a'_7] \cup e'_7 \cup B[b'_7, b_2]$ from $a_1$ to $b_2$ show that $\alpha(A, B) = 2$, a contradiction. \hfill $\Box$

Now, by (2.2.17), $b_1^\# \in B[b_1, b_8]$ and $b_2^\# \in B[b_6, b_1^\#]$. Moreover, we choose $\{b_1^\#, b_2^\#\}$ so that $B[b_1^\#, b_2^\#]$ is maximal. By (2.2.7), $G_0$ contains a path from $a_0$ to $B(b_4, b_6)$, internally disjoint from $B$. Then by Lemma 2.0.8, and by the choice of $\{b'_1, b'_2\}$, we have $b_1^\# = b_1, b_2^\# = b_2$, and one of the following holds:

(N1) $a_0 = a_0^\# = a^\#_0$, and so $c(A, B) \geq 2$;
(N2) \( a_0^\# = a_0, \, b_2^\# \) is a cut vertex of \( G_0 \) separating \( b_2 \) from \( \{a_0, b_1\} \), \( a_0', a_0^\#, b_2^\#, b_2' \) are incident with a common finite face of \( G_0 \), and so \( \alpha(A, B) \leq 1 \);

(N3) \( a_0' = a_0, \, b_1' \) is a cut vertex of \( G_0 \) separating \( b_1 \) from \( \{a_0, b_2\} \), \( a_0', a_0^\#, b_1^\#, b_1' \) are incident with a common finite face of \( G_0 \), and so \( \alpha(A, B) \leq 1 \).

Obviously, by (N1)–(N3), there exists a vertex \( a_0^* \in \{a_0', a_0^\#\} \), such that \( \{b_1', b_2^\#, a_0^*\} \) is a 3-cut in \( G_0 \) separating \( B[b_1', b_2^\#] \) from \( \{a_0, b_1, b_2\} \). We let \( e_9 = a_0b_9 \in \text{E}(G) \) with \( b_9 \in B(b_1', b_2) \) and \( a_9 \in A[a_1, a_2] \), such that \( A[a_1, a_9] \) is minimal.

(2.2.18) There exists \( e_9' = a_0'b_9' \in \text{E}(G) \) with \( a_9' \in A(a_9, a_2) \) and \( b_9' \in B[b_1, b_1'] \cup B(b_2^\#, b_2] \).

For otherwise, \( \{a_0^*, b_1', b_2^\#, a_9, a_2\} \) is a cut in \( G \) separating \( A[a_9, a_2] \cup B[b_1', b_2^\#] \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. \( \Box \)

(2.2.19) \( b_9 \in B(b_1', b_4), a_9 \in A[a_8, a_5], a_9' \in A(a_9, a_5), \) and \( b_9' \in B(b_2^\#, b_7] \).

We first prove that \( a_9 \notin A[a_1, a_8] \). For, suppose \( a_9 \in A[a_1, a_8] \). Then by (2.2.15), \( b_9 \notin B(b_8, b_6) \), and so \( b_9 \in B[b_6, b_2^\#] \), a contradiction to the choice of \( f_1 \).

We claim that \( b_9' \in B(b_2^\#, b_2] \). For, suppose \( b_9' \in B[b_1, b_1'] \). By (2.2.9), \( a_9' \notin A(a_5, a_2] \), and so \( a_9' \in A(a_9, a_5] \), a contradiction to the choice of \( e_8 \).

By (2.2.6), \( a_9' \notin A(a_5, a_2] \), and so \( a_9' \in A(a_9, a_5] \). Furthermore, we have \( b_9' \in B(b_2^\#, b_7] \); or else, \( (e_3, e_4, e_5, e_6, e_9') \) contradicts the choice of \( \mathcal{P} \).

Now, since \( a_9' \in A(a_9, a_5] \), then \( a_9 \neq a_5 \), which implies that \( a_9 \in A[a_8, a_5] \).

Finally, \( b_9 \in B(b_1', b_4] \). First, \( b_9 \notin B[b_5, b_2^\#] \) to avoid the doublecross \( e_9', e_5, e_6, e_9 \). By (2.2.5), \( b_9 \notin B(b_4, b_5] \). So \( b_9 \in B(b_1', b_4] \). \( \Box \)

Now, we choose \( e_9' \) so that \( B[b_2^\#, b_9'] \) is minimal. Since \( a_9' \in A(a_9, a_5] \), then \( a_5 \neq a_9 \).

(2.2.20) (N1) does not hold.
For, suppose (N1) holds. By the choice of \( \{b'_1, b'_2\} \), and by planar structure of \( G_0 \), \( G_0 - B(b'_1, b'_2) - a_0 \) contains a path \( B_1 \) from \( b_1 \) to \( b'_1 \). Moreover, by the choice of \( \{b^*_1, b^*_2\} \), and by planar structure of \( G_0 \), \( G_0 - B(b^*_2, b_2) - a_0 \) contains a path \( B_2 \) from \( b^*_2 \) to \( b_2 \).

We may assume \( G \) has two disjoint edges \( f_8, f_9 \) from \( a^*_8, a^*_9 \in A(a_1, a_8) \) to \( b^*_8, b^*_9 \in B(b'_1, b_2) \), respectively. For otherwise, there exist a vertex \( v \in V(G) \) and a separation \( (G_1, G_2) \) in \( G \), such that \( V(G_1 \cap G_2) = \{b'_1, a_0, b_1, a_1, v, a_8\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(A(a_1, a_8) \cup B(b_1, b'_1)) \subseteq V(G_2) \), and \( (G_2, b'_1, a_0, b_1, a_1, v, a_8) \) is planar, a contradiction to Lemma 2.0.3.

By (2.2.15), \( b^*_8, b^*_9 \in B[b_6, b_2] \). Moreover, by (2.2.10), \( f_8, f_9 \) form a cross. So we may assume \( a_1, a^*_8, a^*_9, a_2 \) occur on \( A \) in order, and \( b_1, b^*_8, b^*_9, b_2 \) occur on \( B \) in order. We further choose \( f_8, f_9 \) with \( a^*_8, a^*_9 \in A(a_1, a_8) \) and \( b^*_8, b^*_9 \in B[b_6, b_2] \) so that \( A[a^*_8, a^*_9] \) is maximal. By the existence of \( e'_9 \) and (2.2.10), we may assume \( b^*_8 \in B(b^*_2, b_2) \).

We claim that \( G \) has an edge \( f_5 \) from \( b^*_8 \in B[b_1, b'_1] \) to \( a^*_9 \in A(a_1, a^*_9) \). For otherwise, all edges from \( B[b_1, b'_1] \) will end in \( \{a_1\} \cup V(A[a^*_8, a_8]) \). By the choice of \( f_8, f_9 \), \( G \) has no edge from \( A(a^*_8, a_8) \) to \( B(b_8, b_2) \). Hence, \( G \) has a separation \( (G_1, G_2) \), such that \( V(G_1 \cap G_2) = \{b'_1, a_0, b_1, a_1, a^*_8, a_8\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(A(a^*_8, a_8) \cup B(b_1, b'_1)) \subseteq V(G_2) \), and \( (G_2, b'_1, a_0, b_1, a_1, a^*_9, a_8) \) is planar. By Lemma 2.0.3, \( |V(G_2 - G_1)| = 1 \). So \( V(G_2 - G_1) = \{b_5\} \), and \( G \) has edges from \( b_8 \) to \( b'_1, a_0, b_1, a_1, a^*_9, a_8 \), respectively. But then, \( b_1 \) has degree 1 in \( G \), a contradiction.

By (2.2.7), there exists a path \( A_0 \) from \( a_0 \) to \( B(b_4, b_6) \) in \( G_0 \), internally disjoint from \( B \).

Now, combined with Lemma 3.0.1, the path \( B_1 \cup B[b'_1, b_9] \cup e_9 \cup A[a^*_8, a_9] \cup f_9 \cup B[b^*_9, b^*_2] \cup B_2 \) from \( b_1 \) to \( b_2 \), the path \( B[b_1, b^*_9] \cup f_5 \cup A[a^*_9, a^*_8] \cup f_8 \cup B[b^*_9, b_2] \) from \( b_1 \) to \( b_2 \), and the path \( A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2] \) from \( a_0 \) to \( a_2 \) show that \( \alpha(A, B) = 2 \) and \( c(A, B) = 0 \), a contradiction. □

(2.2.21) (N2) does not hold.

For, suppose (N2) holds. We may assume \( G \) has an edge \( e''_7 \) from \( a''_7 \in A(a_1, a_8) \) to \( b''_7 \in B(b'_1, b_2) \), such that \( e''_7 \cap e'_7 = \emptyset \). For otherwise, by (2.2.1), (2.2.10) and (2.2.15), \( G \) has a
separation \((G_1, G_2)\), such that \(V(G_1 \cap G_2) = \{v, a_8, b'_1, a'_0\}\) with \(v \in \{a'_7, b'_7\}\), \(a_0, a_1, b_1 \in V(G_2), |V(G_2 - G_1)| \geq 4, a_2, b_2 \in V(G_1)\), and \((G_2, a_0, b_1, a_1, v, a_8, b'_1, a'_0)\) is planar. Now, it contradicts Lemma 2.0.3 (when \(v = a'_7 = a_1\)) or Lemma 2.0.4 (when \(v \neq a_1\)).

By (2.2.10) and (2.2.15), \(a''_7 \in A(a'_7, a_8)\) and \(b''_7 \in B[b_6, b'_7]\). Now, we choose \(e''_7\) so that \(A[a_1, a''_7]\) is maximal. Then we may assume \(a''_7 \in A(a', a_8)\). For otherwise, \(a''_7 \in A[a_1, a']\), and so \(G\) has no edge from \(A(a', a_8)\) to \(B(b'_1, b_2)\) by the choice of \(e''_7\). But then, \(G\) has a separation \((G_1, G_2)\), such that \(V(G_1 \cap G_2) = \{a', a_8, b'_1, a'_0, b_1\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)\), and \((G_2, a', a_8, b'_1, a'_0, a_0, b_1)\) is planar, a contradiction to Lemma 2.0.3.

By the choice of \(\{a''_0, b''_1, b''_2\}\), and by planar structure of \(G_0\), we may assume \(G_0 - B[b'_7, b_2]\) contains a path \(B_2\) from \(b_2\) to \(b''_2\).

Now, let \(A_0\) be the path from \(a_0\) to \(B(b_4, b_6)\) in \(G_0\), internally disjoint from \(B\). Moreover, we further choose \(A_0\) such that \(A_0[a_0, a'_0]\) is on the boundary of \(G_0\) without going through \(b_1\).

We claim that \(G_0 - B(b_1, b') - A_0\) contains a path \(B_1\) from \(b_1\) to \(b'_1\). For otherwise, \(b'_1 \neq b_1\), and there exist a vertex \(v_1 \in A_0[a_0, a'_0]\) and a vertex \(v_2 \in B(b_1, b')\), such that \(v_1, v_2\) are incident with a common finite face of \(G_0\). Now, combined with (2.2.12), if \(v_1 \neq a_0\), then \(\{b_1, v_1, v_2, b_2\}\) is a cut in \(G\) separating \(a_0\) from \(\{a_1, a_2\}\), a contradiction; if \(v_1 = a_0\), then \(\{v_1, v_2, b_1\}\) is a cut in \(G\) separating \(N_G(b_1)\) from \(\{a_0, a_1, a_2, b_1, b_2\}\), a contradiction.

Now, combined with Lemma 3.0.1, the path \(B_1 \cup B[b'_1, b_9] \cup e_9 \cup A[a''_7, a_9] \cup e''_7 \cup B[b''_7, b''_2] \cup B_2\) from \(b_1\) to \(b_2\), the path \(A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2]\) from \(a_0\) to \(a_2\), the path \(A[a_1, a'_0] \cup e' \cup B[b_1, b']\) from \(a_1\) to \(b_1\), and the path \(A[a_1, a'_7] \cup e'_7 \cup B[b'_7, b_2]\) show that \(\alpha(A, B) = 2\), a contradiction. \(\square\)

Hence, (N3) holds. We may assume \(G\) has an edge \(e''_7\) from \(a''_7 \in A(a', a_8)\) to \(b''_7 \in B(b'_1, b_2)\), such that \(e''_7 \cap e'_7 = \emptyset\). For otherwise, by (2.2.10) and (2.2.15), \(G\) has a separation \((G_1, G_2)\), such that \(V(G_1 \cap G_2) = \{v, a', a_8, b_1, b'_1\}\) with \(v \in \{a'_7, b'_7\}\), \(V(A[a', a_8] \cup B[b_1, b'_1]) \subseteq V(G_1)\), and \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_2)\), a contradiction.

By (2.2.10) and (2.2.15), we may also assume \(a''_7 \in A(a'_7, a_8)\) and \(b''_7 \in B[b_6, b'_7]\). By
the choice of \( \{a'_0, b'_1, b'_2\} \), and by planar structure of \( G_0 \), we may assume \( G_0 - B(b_1, b') \) contains a path \( B_1 \) from \( b_1 \) to \( b'_1 \).

Now, let \( A_0 \) be the path from \( a_0 \) to \( B(b_4, b_6) \) in \( G_0 \), internally disjoint from \( B \). Moreover, we further choose \( A_0 \) such that \( A_0[a_0, a_0^#] \) is on the boundary of \( G_0 \) without going through \( b_2 \).

We claim that \( G_0 - B[b'_7, b_2] - A_0 \) contains a path \( B_2 \) from \( b_2 \) to \( b'_2 \). For otherwise, \( b'_7 \neq b_2 \), and there exist a vertex \( v_1 \in A_0[a_0, a_0^#] \) and a vertex \( v_2 \in B[b'_7, b_2] \), such that \( v_1, v_2 \) are incident with a common finite face of \( G_0 \). Now, combined with (2.2.11), if \( v_1 \neq a_0 \), then \( \{b_1, v_1, v_2, b_2\} \) is a cut in \( G \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction; if \( v_1 = a_0 \), then \( \{v_1, v_2, b_2\} \) is a cut in \( G \) separating \( N_{G}(b_2) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Now, combined with Lemma 3.0.1, the path \( B_1 \cup B[b'_7, b_2] \cup e_9 \cup A[a'_7, a_9] \cup e'_7 \cup B[b''_7, b'_2] \cup B_2 \) from \( b_1 \) to \( b_2 \), the path \( A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2] \) from \( a_0 \) to \( a_2 \), the path \( A[a_1, a'_7] \cup e' \cup B[b_1, b'] \) from \( a_1 \) to \( b_1 \), and the path \( A[a_1, a'_7] \cup e'_7 \cup B[b'_7, b_2] \) show that \( \alpha(A, B) = 2 \), a contradiction. \( \square \)
CHAPTER 7
FUTURE WORK

7.0.1 A characterization of two-three linked graphs

In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. Here, we believe we have such a characterization, although it is quite complicated (even to state) and its proof is longer.

We say that \((G, a_0, a_1, a_2, b_1, b_2)\) is **reducible**, if one of the following holds:

(R1) \(G\) has an edge \(e\) with one end in \(\{a_0, a_1, a_2\}\) and one end in \(\{b_1, b_2\}\).

(R2) There exists a separation \((G_1, G_2)\) in \(G\) of order at most 1.

(R3) There exists a separation \((G_1, G_2)\) in \(G\) of order 2, satisfying one of the following properties:

(a) \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)\) and \(V(G_2 - G_1) \neq \emptyset\); or

(b) \(|V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}| = 1\) and \(|E(G_2)| \geq 3\); or

(c) for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2\}\), \(a_i, b_j \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)\), and \((G_2, a_i, b_j, c_2, c_1)\) is planar; or

(d) for some \(j \in \{1, 2\}\) and some permutation \(\pi\) of \(\{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2\}, a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1), a_{\pi(2)}, b_{3-j} \in V(G_1)\), and \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_2, c_1)\) is planar; or

(e) for some \(i \in \{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2\}\), \(a_i, b_1, b_2 \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_1, b_2\} \subseteq V(G_1)\), and \((G_2, b_1, a_i, b_2, c_2, c_1)\) is planar.
(R4) There exists a separation \((G_1, G_2)\) in \(G\) of order 3, satisfying one of the following properties:

(a) \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)\) and \(V(G_2 - G_1) \neq \emptyset\); or

(b) \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(\{d\} = \{a_0, a_1, a_2, b_1, b_2\} \cap V(G_2 - G_1)\), \((G_2, d, c_3, c_2, c_1)\) is planar, and \(|V(G_2 - G_1)| \geq 2\); or

(c) for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(a_i, b_j \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)\), \((G_2, a_i, b_j; c_1, c_2, c_3)\) is planar, and \(|V(G_2 - G_1)| \geq 3\); or

(d) for some permutation \(\pi\) of \(\{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(a_\pi(0), a_\pi(1), b_j \in V(G_2 - G_1)\), \(a_\pi(2), b_{3-j} \in V(G_1)\), and \((G_2, a_\pi(0), b_j, a_\pi(1), c_3, c_2, c_1)\) is planar; or

(e) for some \(i \in \{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(b_1, a_i, b_2 \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2\} - \{a_i\} \subseteq V(G_1)\), and \((G_2, b_1, a_i, b_2, c_3, c_2, c_1)\) is planar.

(R5) There exists a separation \((G_1, G_2)\) in \(G\) of order 4, satisfying one of the following properties:

(a) let \(W\) be a graph with \(V(W) = \{w_0, w_1, w_2, w_3, w_4\}\), \(E(W) = \{w_0 w_i; i = 1, 2, 3, 4\} \cup \{w_1 w_2, w_1 w_3\}\), then \(a_0, a_1, a_2, b_1, b_2 \in V(G_1), V(G_2 - G_1) \neq \emptyset\), and \(G_2\) is not a subgraph of \(W\); or

(b) \(V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}\), \(a_0, a_1, a_2, b_1, b_2 \in V(G_1), V(G_2 - G_1) = \{c\}\), \(G\) has edges from \(c\) to \(c_1, c_2, c_3, c_4\), \(G\) has edges from \(c_1\) to \(c_2, c_3\), and for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(a_i, b_j \in V(G_1 \cap G_2)\); or

(c) for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, a_i, b_j\}\), \(a_0, a_1, a_2, b_1, b_2 \in V(G_1), V(G_2 - G_1) = \{c\}\), \(G\) has edges from \(c\) to \(c_1, c_2, a_i, b_j\), and \(G\) has an edge from \(c_1\) to \(c_2\); or
(d) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) = \{c\}$, $G$ has edges from $c$ to $c_1, c_2, c_3, c_4$, $G$ has an edge from $c_1$ to $c_2$, and for some permutation $\pi$ of $\{0, 1, 2\}$, $\{a_{\pi(0)}, a_{\pi(1)}\} \subseteq V(G_1 \cap G_2)$ and $\{a_{\pi(0)}, a_{\pi(1)}\} \cap \{c_1, c_2\} \neq \emptyset$; or

(e) for some $i \in \{0, 1, 2\}$, $\{a_{i}\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{b_1, b_2, c_1, c_2\}$, $(G_2, a_1, b_1, c_1, c_2, b_2)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(f) for some permutation $\pi$ of $\{0, 1, 2\}$ and some $j \in \{1, 2\}$, $\{b_{j}\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{a_{\pi(1)}, a_{\pi(2)}, c_1, c_2\}$, $(G_2, b_{j}, a_{\pi(1)}, c_1, c_2, a_{\pi(2)})$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(g) for some permutation $\pi$ of $\{0, 1, 2\}$ and some $j \in \{1, 2\}$, $\{a_{\pi(0)}\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{a_{\pi(1)}, c_1, c_2\}$, $(G_2, a_{\pi(0)}, b_{j}, c_1, a_{\pi(1)}, c_2)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(h) for some permutation $\pi$ of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(0)}\}$, $a_{\pi(1)}, b_{j} \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, c_1, c_2, a_{\pi(0)}, c_3, a_{\pi(1)}, b_{j})$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(i) for some permutation $\pi$ of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(0)}\}$, $a_{\pi(1)}, b_{j} \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, a_{\pi(0)}, b_{j}, a_{\pi(1)}, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(j) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, b_{j}\}$, $a_{i}, b_{3-j} \in V(G_2 - G_1)$, $\{a_1, a_2, a_3\} - a_{i} \subseteq V(G_1)$, $(G_2, b_{3-j}, a_{i}, b_{j}, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(k) for some permutation $\pi$ of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_{\pi(0)}, a_{\pi(1)}, b_{j} \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, a_{\pi(0)}, b_{j}, a_{\pi(1)}, c_4, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 4$; or

(l) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_{i}, b_{1}, b_{2} \in V(G_2 - G_1)$, $\{a_1, a_2, a_3\} - a_{i} \subseteq V(G_1)$, $(G_2, b_{1}, a_{i}, b_{2}, c_4, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 4$; or
(m) for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( a_{\pi(0)}, a_{\pi(1)}, b_1, b_2 \in V(G_1), \{a_{\pi(0)}, a_{\pi(1)}, b_1, b_2\} \cap V(G_2) \neq \emptyset, a_{\pi(2)} \in V(G_2) - V(G_1) \), and \( G_1 \) has a disk representation in which \( a_{\pi(0)}, b_1, a_{\pi(1)}, b_2 \) occur on the boundary of the disk in the order listed and the vertices in \( V(G_1) \cap V(G_2) \) are incident with a common finite face.

(R6) There exists a separation \((G_1, G_2)\) in \( G \) of order 5, satisfying one of the following properties:

(a) \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), E(G[\{c_1, c_2, c_3, c_4, c_5\}]) \subseteq E(G_1), (G_2, c_1, c_2, c_3, c_4, c_5) \) is planar, and \(|V(G_2 - G_1)| \geq 2\); or

(b) \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), \) and for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( G_1 \) has a disk representation with the vertices \( a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)} \) drawn on the boundary of the disk in the order listed; or

(c) for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}, a_{\pi(0)} \in V(G_1 - G_2), a_{\pi(0)} \in V(G_2 - G_1), (G_2, b_1, c_1, a_{\pi(1)}, c_2, b_2, a_{\pi(0)}) \) is planar, and \(|V(G_2 - G_1)| \geq 4\); or

(d) for some \( j \in \{1, 2\} \) and some permutation \( \pi \) of \( \{0, 1, 2\} \), \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(1)}, b_j\}, a_{\pi(2)} \in V(G_1 - G_2), a_{\pi(0)}, b_{3-j} \in V(G_2 - G_1), (G_2, a_{\pi(1)}, c_1, c_2, c_3, b_j, a_{\pi(0)}, b_{3-j}) \) is planar, and \(|V(G_2 - G_1)| \geq 3\).

Actually, we can prove that if \((G, a_0, a_1, a_2, b_1, b_2)\) is reducible, then we could either easily determine whether or not \((G, a_0, a_1, a_2, b_1, b_2)\) is feasible, or reduce \((G, a_0, a_1, a_2, b_1, b_2)\) to \((G', a'_0, a'_1, a'_2, b'_1, b'_2)\) with \(|V(G)|, |E(G)| > (|V(G')|, |E(G')|)\) in lexicographic order, such that \((G, a_0, a_1, a_2, b_1, b_2)\) is feasible iff \((G', a'_0, a'_1, a'_2, b'_1, b'_2)\) is feasible.

With all these, we can state our main result.

**Theorem 7.0.1** Let \((G, a_0, a_1, a_2, b_1, b_2)\) be a rooted graph. Then one of the following conclusions holds:
There exists a cluster \( \{ X_1, X_2 \} \) in \( G \) such that \( \{ a_0, a_1, a_2 \} \subseteq X_1 \) and \( \{ b_1, b_2 \} \subseteq X_2 \).

\( (G, a_0, a_1, a_2, b_1, b_2) \) is reducible.

For some \( i \in \{ 0, 1, 2 \} \), \( G - a_i \) has no cluster \( \{ X_1, X_2 \} \) such that \( \{ a_0, a_1, a_2 \} - \{ a_i \} \subseteq X_1 \) and \( \{ b_1, b_2 \} \subseteq X_2 \).

There exist a permutation \( \pi \) of \( \{ 0, 1, 2 \} \), a graph \( H \) and vertices \( s, t, s', t' \in V(H) \) such that \( G \) is obtained from \( H \) by identifying \( s \) with \( s' \) and \( t \) with \( t' \), respectively, and \( H \) has a disk representation with the vertices \( a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)}, s, t, s', t' \) drawn on the boundary of the disk in the order listed.

\( G \) has a separation \( (G_1, G_2) \) in \( G \) of order 4, such that \( V(G_1 \cap G_2) = \{ c_1, c_2, c_3, c_4 \} \), \( a_0, a_1, a_2, b_1, b_2 \in V(G_1) \), and there exist a permutation \( \pi \) of \( \{ 0, 1, 2 \} \), a graph \( H \) and vertices \( c'_2, c''_2 \in V(H) \), where \( G_1 \) is obtained from \( H \) by identifying \( c'_2 \) with \( c''_2 \), \( (H, a_{\pi(1)}, b_1, a_{\pi(0)}, b_2, a_{\pi(2)}, c''_2, c_4, c_3, c'_2, c_1) \) is planar, and \( c_2 \in V(G_1) \) is the vertex obtained by identifying \( c'_2 \) with \( c''_2 \).

7.0.2 Clarifying (C3)

Note that if (C4) or (C5) holds, then (C1) will not hold. However, if (C3) holds, \( (G, a_0, a_1, a_2, b_1, b_2) \) may be feasible or may be infeasible. Although by using 2-linkage algorithms, it is easy to judge whether \( (G, a_0, a_1, a_2, b_1, b_2) \) admits (C3), we want to give a more precise characterization of feasible rooted graphs when (C3) holds.

We will still assume \( G \) is not reducible. So by applying Seymour’s version of 2-linkage theorem in [37], when (C3) holds, there exists \( i \in \{ 0, 1, 2 \} \), such that \( (G - a_i, a_{i+1}, b_1, a_{i-1}, b_2) \) is planar. So \( G \) actually is an apex graph.

7.0.3 A faster algorithm

Another possible future work is to develop a faster polynomial time algorithm for the Two-Three Linkage Problem.
Note that the existence of such an algorithm with polynomial running time is guaranteed by the work of Robertson and Seymour in [40]: Given a graph \( G \) and \( k \geq 1 \) pairs of vertices \( \{s_i, t_i\}, i = 1, \ldots, k \) of \( G \) with \( k \) fixed, there exists a polynomial time algorithm for deciding if there are \( k \) mutually internally vertex-disjoint paths in \( G \) joining \( s_i \) and \( t_i \), \( i = 1, \ldots, k \). In fact, to resolve the Two-Three Linkage Problem, we just need to check:

(i) whether for some \( i \in \{0, 1, 2\} \), \( G \) contains 3 mutually internally vertex-disjoint paths joining the pairs \( \{b_1, b_2\}, \{a_{i-1}, a_i\} \) and \( \{a_i, a_{i+1}\} \); or

(ii) whether for some vertex \( v \in V(G) - \{a_0, a_1, a_2, b_1, b_2\} \), \( G \) contains 4 mutually vertex-disjoint paths to join the pairs \( \{b_1, b_2\}, \{v, a_0\}, \{v, a_1\} \) and \( \{v, a_2\} \).

Clearly, the answer is yes iff \( (G, a_0, a_1, a_2, b_1, b_2) \) is feasible. The disjoint paths algorithm of Robertson and Seymour has running time \( O(|V(G)|^3) \). So the above algorithm runs \( O(|V(G)|^4) \) time.

However, the disjoint paths algorithm of Robertson and Seymour is not practical, since it involves an enormous constant. Hence, it is meaningful to come up with a faster algorithm for the two-three linkage problem. In fact, to the best of our knowledge, Tholey [41] found the \( O(m + n\alpha(n, n)) \)-time algorithm, the currently best known nearly linear time bound, of 2-linkage problem, where \( \alpha \) denotes the inverse of the Ackermann function. By repeatedly using 2-linkage algorithm, we expect to obtain a \( O(|V(G)|^3) \)-time two-three linkage algorithm.

### 7.0.4 Related conjecture

A graph \( G \) is apex if \( G - v \) is planar for some vertex \( v \in V(G) \). Jørgensen [34] conjectured that every 6-connected graph with no \( K_6 \)-minor is apex.

In the two-three linkage problem, we only consider finding disjoint connected subgraphs \( G_1, G_2 \) such that \( \{a_0, a_1, a_2\} \subseteq V(G_1) \) and \( \{b_1, b_2\} \subseteq V(G_2) \). However, it is also natural to ask whether we can find such disjoint connected subgraphs \( G_1, G_2 \) satisfying...
additional properties. For example, we have

**Conjecture 7.0.2** Any 6-connected non-apex graph $G$ with distinct vertices $a_0, a_1, a_2, b_1, b_2 \in V(G)$ contains disjoint connected subgraphs $G_1, G_2$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1), \{b_1, b_2\} \subseteq V(G_2)$, and the following properties hold:

(P1) there exists a vertex $v \in G_1 - \{a_0, a_1, a_2\}$ such that $G_1$ has three disjoint paths from $v$ to $a_0, a_1, a_2$, respectively;

(P2) for each vertex $v \in G_1$, the vertices $a_0, a_1, a_2$ are contained in one component of $G_1 - v$.

One observation is that if there exists $v \in V(G)$ such that $(G-v, a_1, b_1, a_2, b_2)$ is planar, then there do not exist disjoint connected subgraphs $G_1, G_2$ in $G$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1), \{b_1, b_2\} \subseteq V(G_2)$, and $G_1$ satisfies (P1) and (P2). Note that such $G$ is apex, and $G$ can be 6-connected.

If Conjecture 7.0.2 is true, we may prove that given a 6-connected graph $G$ and triangles $a_ib_jb_i$ for $i = 0, 1, 2$, $G - b_ib_j - \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ contains disjoint connected subgraphs $G_1, G_2$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1), \{b_1, b_2\} \subseteq V(G_2)$, and $G_1$ satisfies (P1) and (P2). Such properties could be useful in resolving Jørgensen’s conjecture for 6-connected graph in which some edge is contained in three triangles.
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